

Math. P. 63ⁿ

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DUAL ARITHMETIC.

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DUAL ARITHMETIC.

A NEW ART.

INVENTED AND DEVELOPED

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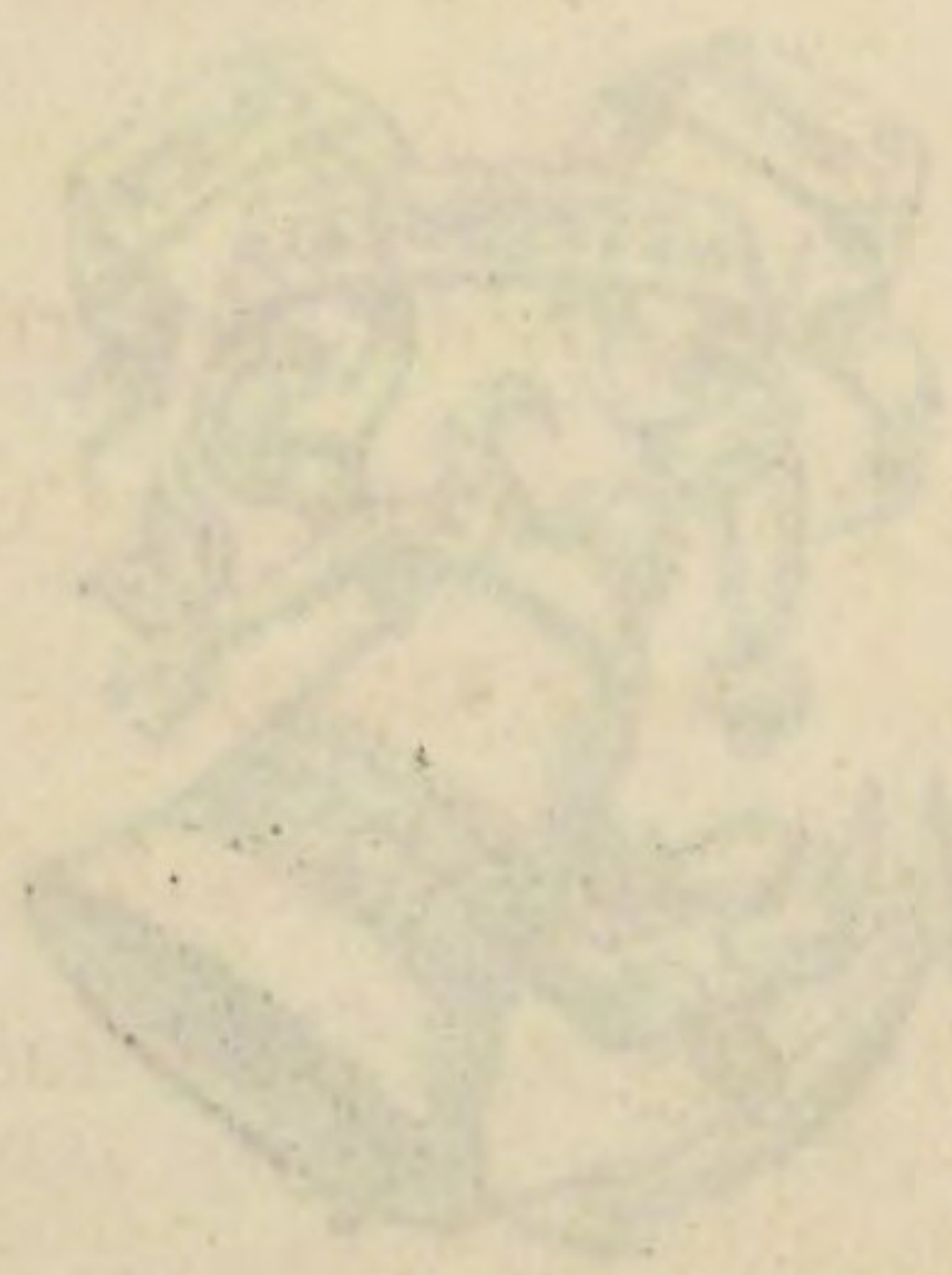
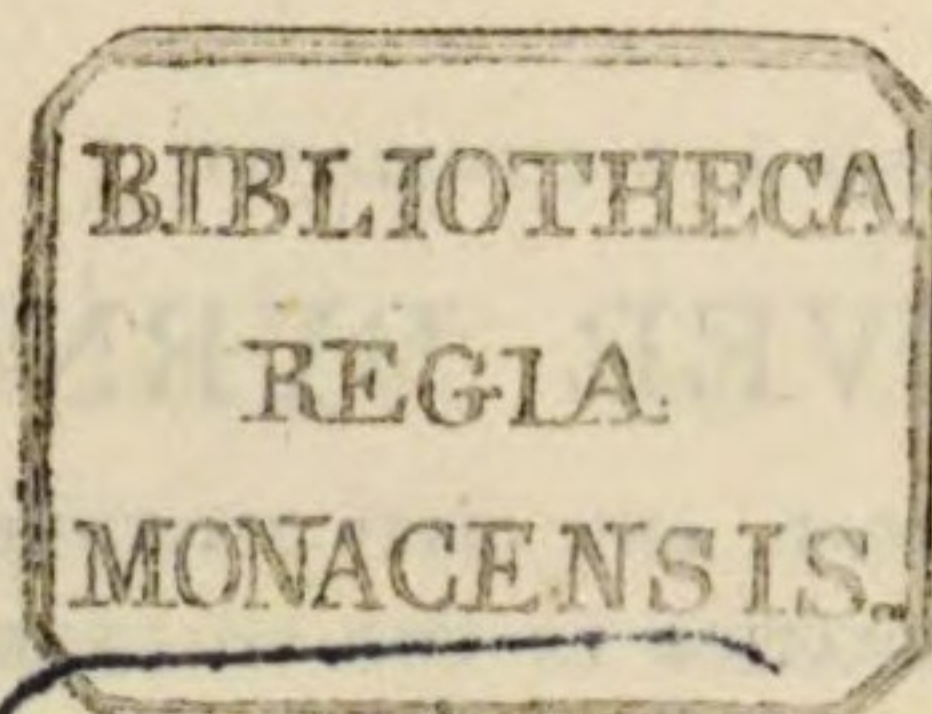


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PREFACE.

IN the most general sense, the peculiar system of developing and applying the power of numbers, which I have called DUAL ARITHMETIC, is a new ART, and not merely a new method of obtaining results that might be found by arts previously known.

Dual Arithmetic unfolds the capabilities of numbers in an original manner, extends the boundaries of mathematical science, and establishes new rules by which many difficult problems of the greatest utility and importance are solved with ease without the aid of tables, cumbersome formulæ, or methods of approximation.

Those acquainted with the operations of common arithmetic can easily acquire a sufficient knowledge of Dual Arithmetic to understand the solutions of the following introductory examples.

In developing the elements of this new art, I have purposely disregarded the most obvious arithmetical abridgments, in order that each process may appear without the least disguise.

The Introductory Examples are for the purpose of showing the accomplished mathematician the power of this art. The learner who wishes to acquire it practically, is recommended to pass them over until he has mastered the work, when he will be able fully to appreciate their value.

I have applied the numerical operations of Dual Arithmetic to a variety of developments, so that in each particular inquiry the best method to connect actual calculation with algebraic language and symbolical expressions may be applied. In my works on Algebra and the Calculus, which are being prepared for publication, the whole subject and its different applications will be treated in a general and exhaustive manner. My work on the Calculus, to be termed the "Calculus of Form," unfolds a new science and establishes modern analysis on a purely Mathematical basis, rejecting the reasoning of the Differential and other methods.

OLIVER BYRNE.

CONTENTS.

INTRODUCTORY EXAMPLES.

PART I.

DEFINITIONS AND ELEMENTARY PROPOSITIONS.

Why this work is entitled "Dual Arithmetic," p. 1.

PROPOSITION I.

To multiply a given number by a given power of 1'1, 1'01, 1'001, &c. p. 2.
—Coefficients of $(x + y)^n$, when expanded, n being a whole positive number, p. 2.—Examples: Periods of figures corresponding to particular multipliers, p. 3.—Properties of particular multipliers, p. 4.—Different methods of obtaining results, without interfering with the laws that govern the system, p. 5.—The use of blank periods of dots, p. 7.—The office of the new sign $\downarrow$, defined, p. 9.—Arrangements to save mental labour, pp. 10—15.

PROPOSITION II.

To find the powers of (1'1), (1'01), (1'001), &c., so that when one given number is multiplied by them, the continued product will be another given number, p. 15.—Examples: How to find the operating figures to the right of $\downarrow$, &c., p. 16.—Reduction of compound fractions to the powers of $\downarrow 1$, $\downarrow 0, 1$, &c., p. 17.

PART II.

DIVISION OF DUAL ARITHMETIC, p. 21.

The operative numbers employed in division, p. 22.—The nature of negative numbers to the right of $\downarrow$; Examples, &c., p. 23.—Division of

Dual Arithmetic proved by multiplication, p. 25.—Numbers employed in division, with their application to particular examples, p. 26.—A method of division independent of the rules of Common Arithmetic, p. 28.

PART III.

EVOLUTION AND INVOLUTION OF DUAL ARITHMETIC p. 29.

How to find, to any proposed extent, the relations existing between $\downarrow 1$, $\downarrow 0,1$, $\downarrow 0,0,1$, &c., p. 31.—Conditional equalities in a tabulated form; verifications, examples, and comparisons, p. 32.—Extraction of the roots of given numbers; Examples and comparisons, p. 36.—Values of $\downarrow 1$, $\downarrow 0,1$, $\downarrow 0,0,1$, &c., in a tabulated form, p. 42.—To find the value of x in the equation $(1.251853)^x = 1.571653$, with other examples of the same order, p. 43.—To find the hyperbolic log. of $\pi = 3.14159$, &c., p. 48.—Values of $\downarrow 1$, $\downarrow 0,1$, $\downarrow 0,0,1$, &c., under different forms of arrangement, p. 51.—Values of 2, 3, 4, &c., reduced to the eighth position; how the reduction is effected, p. 54.

PART IV.

OF ANGULAR MAGNITUDES AND TRIGONOMETRICAL LINES, p. 59.

To find the sine and cosine of a given angle, p. 60.—Lengths of arcs to radius 1, p. 62.—Examples of transformations of the equations of condition, p. 63.—The determination of angular magnitudes and trigonometrical lines by the combined operations of both Common and Dual Arithmetic, p. 67.—Table, to find the length of an arc of any given number of degrees, minutes, &c., p. 72.—A new and easy method by which the moon's distance may be cleared, so as to find the longitude at sea, p. 75.—The cosine of an angle being given, to find the degrees, minutes, &c., contained in the arc, p. 79.—To find the degrees, &c., contained in a given arc without the use of tables, p. 81.—The use of formulæ of verification, the operations of Common and Dual Arithmetic combined, p. 83.

PART V.

FORMULÆ AND RULES FOR THE SOLUTION OF PLANE TRIANGLES, WITHOUT THE USE OF TABLES, p. 87.

Calculations combining Common and Dual Arithmetic, p. 88.—How to

operate upon quantities, by Dual Arithmetic only, p. 89.—Particular examples in the solution of plane triangles, p. 93.—Given the three sides of a plane triangle to find the area and one of the angles, independent of tables and Common Arithmetic, p. 105.

PART VI.

DUAL ARITHMETIC APPLIED TO THE SOLUTION OF THE DIFFERENT CASES OF SPHERICAL TRIANGLES.

Right-angled spherical triangles; Examples, p. 110.—Oblique spherical triangles; Dual and Common Arithmetic combined, p. 115.—The areas of spherical triangles, and the spherical excess, p. 117.—An example solved by two methods, each independent of Common Arithmetic, p. 120.

IMPORTANT MISCELLANEOUS PROBLEMS SOLVED BY DUAL ARITHMETIC.

To find the hyperbolic logarithm of a given number, p. 123.—Unabridged examples; how to find a quotient by simple subtraction, p. 125.—How to avoid large numbers to the right of $\downarrow$, in general developments, p. 126.—To find the hyperbolic logarithm of a number greater than 10, p. 128.—To find the number corresponding to a given hyperbolic logarithm, p. 132.—To find the distance of an inaccessible object, p. 134.—To find the meridional parts answering to any given latitude by direct calculation, p. 142.—To find the units of work done on each square inch of the piston of a steam engine, the steam being worked expansively, p. 147.—To find the time of oscillation of a simple circular pendulum, p. 150.—By the direct application of Dual Arithmetic, and without the direct use of the well-known constant π , to find the length of an arc corresponding to any number of degrees, &c., p. 153.—To clear the moon's distance so as to find the longitude at sea, p. 154.—To find the length of an elliptic arc, investigation, and numerical example, p. 156.—To determine the numerical value of elliptic and hyperbolic functions when integrated, p. 161.—To find the meridional parts corresponding to latitude $83^\circ 25' 24''$, p. 167.—Without the aid of Common Arithmetic, to find the value of $\frac{1.3.5.7.9.11.13}{2.4.6.8.10.12.14} x^7$, a term in the series that expresses the time of oscillation of a circular pendulum, when $x = .113574657$, p. 169.—Given the cosine of an angle, to find the sine, tangent, cotangent, &c., and their hyperbolic and common logarithms, p. 170.

GENERAL SOLUTION OF ALGEBRAICAL EQUATIONS OF ALL DEGREES, p. 173

To multiply a number by $\downarrow 5$, $\downarrow 0,5$, $\downarrow 0,0,5$, &c. Monogram by which the positions may be marked; examples, p. 173.—To extract roots of cubic equations, p. 176.—To extract roots of an equation of the fourth degree, p. 177.—To extract roots of equations of the fifth degree, p. 179.—To extract roots of equations of the sixth and higher degrees, p. 185.—To extract the roots of equations containing fractional indices, p. 190.—Exponential equations, p. 194.—To find the coordinates of the curve $y = \frac{2}{\sqrt{\pi}} e^{-x^2}$, when the area is the fraction $\frac{2143}{4789}$, p. 201.—Given, $x \sin x = \cos x$, to find x , p. 203.

GENERAL DEVELOPMENTS.

General development of any quantity or magnitude capable of being represented by numbers, p. 206.—Ultimate values of $\downarrow 1$, $\downarrow 0,1$, $\downarrow 0,0,1$, &c., p. 212.—Log. $\pi = 16\frac{5}{9} + \log. \frac{2}{3}$, p. 215.—Developments with operative numbers on the right and left of $\downarrow$, p. 216.

APPENDIX.

Notes to the Introductory Examples p. 217

INTRODUCTORY EXAMPLE

The value of the function $f(x)$ is given by the following table:

x	$f(x)$
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100

Required: the value of the function $f(x)$ at $x = 10$.

Answer: $f(10) = 100$.

Then by means of the following table we can find the value of the function $f(x)$ at $x = 10$.

Table 1. The value of the function $f(x)$ at $x = 10$.

The value of the function $f(x)$ is given by the following table:

Table 2. The value of the function $f(x)$ at $x = 10$.

ERRATA.

Page 5, line 5, from top, *for* " $54.247 \times (1.06)$," *read* " $54.247 \times (1.01)$."

„ 21, „ 6, „	„	"divide by 1, $(1.01)^3$," <i>read</i> "divide 1 by $(1.01)^3$."
„ 22, „ 19, „	„	"negative," <i>read</i> "positive."
„ 22, „ 20, „	„	"positive," <i>read</i> "negative."
„ 38, „ 11, „	„	"sixth root," <i>read</i> "fifth root."
„ 72, „ 14, „	„	"minutes," <i>read</i> "minutes, &c."
„ 117, „ 9, „	„	"THE AREA," <i>read</i> "THE AREAS."

INTRODUCTORY EXAMPLES.

1. *The radius of the earth at the equator = 20921185 feet ;
and the area of circle whose diameter is unity = 78539816.*

Required the value of $\left(\frac{78539816}{20921185}\right)^{\frac{1}{2}}$

$$78539816 = 10^4 \times 7 \downarrow 1,1,9,8,5,4,4,7, = \downarrow 1127192741,$$

$$20921185 = 10^6 \times 2 \downarrow 0,4,5,2,3,1,2,0, = \downarrow 1455441914,$$

$$\begin{array}{r} + 1127192741 \\ - 1455441914 \\ \hline 7 \) - 328249173 \end{array}$$

$$- 46892739 = \frac{6}{10} \downarrow 0,4,2,1,2,1,4,8, = \cdot 625687758.$$

2. *Required the hyperbolic logarithm of $3 \cdot 141592654 = \pi$, or
find the value of x in the equation*

$$\varepsilon^x = \pi ;$$

$$\varepsilon = 2 \cdot 718281828.$$

$$\varepsilon = \downarrow 10,4,7,1,0,0,3,8, = \downarrow 100005025,$$

$$\pi = \downarrow 12,0,1,0,0,8,2,3, = \downarrow 114478742,$$

Then by common division 114478742 divided by 100005025, gives 1.14472989, the hyperbolic logarithm of π .

The divisor 100005025, which is a whole number, remains constant in calculating hyperbolic logarithms. The division by 100005025 = $\downarrow 0,0,0,0,5,0,2,5, = \downarrow 0,0,0,0,5,0,0,25,$ may be performed by Dual Arithmetic as follows :

$$114478742 \div \downarrow 0,0,0,0,5,0,0,25, = 114478742 \downarrow 0,0,0,0,5,0,0,25,$$

$$\begin{array}{r} 11447 \overline{) 8742} . + \\ 5724 . - \end{array}$$

$$\begin{array}{r} 11447 \overline{) 3018} \dots\dots\dots + \\ 29 \dots\dots\dots - \end{array}$$

$$1.1447 \, 298 \, 9$$

3. Required the hyperbolic logarithm of 1.25, retaining the common divisor 100005025, or multiplier $\downarrow 0,0,0,0,5,0,0,25,$

$$1.25 = \downarrow 2,3,2,6,7,3,2,3, = \downarrow \overline{22315476},$$

$$\begin{array}{r} 22315 \overline{) 476} \dots \\ 1116 \dots \end{array}$$

$$\begin{array}{r} 22314 \overline{) 360} \dots\dots\dots \\ 6 \dots\dots\dots \end{array}$$

$$\text{Hyp. log. } 1.25 = .22314 \, 354$$

4. Find the logarithm of $\pi = 3.141592654$, or the value of x in the equation,

$$10^x = 3.141592654$$

$$10 = \downarrow 24,1,5,1,9,2,9,5, = \downarrow \overline{230270081},$$

$$\pi = \downarrow 12,0,1,0,0,8,2,3, = \downarrow \overline{114478742},$$

Then, by common division, 114478742 divided by 230270081 gives .49714987 the logarithm of π .

In calculating common logarithms by this new art, 230270081 = 23 $\downarrow 0,0,1,1,7,4,0,8$ is constant, and may be retained, and the division performed thus,

$$497 \times 23 = 11431$$

$$11431 \downarrow 0,0,1,4,7,6,5,1, = 114478742$$

$$\therefore \frac{497 \times 23 \downarrow 0,0,1,4,7,5,6,1,}{23 \downarrow 0,0,1,1,7,4,0,8,} = 497 \downarrow 0,0,0,3,0,1,5,3, = .497149876.$$

5. *The distance of the earth from the sun = 95364768 miles, find the common logarithm of this number, or solve the equation,*

$$10^x = 95364768$$

$$9 \overline{) 95364768}$$

$$10596085 \cdot 33 = \downarrow 0,5,8,1,5,1,8,8,$$

$$\downarrow 0,5, = 4975415$$

$$0,0,8, = 799640$$

$$\text{and, } 15188$$

$$9. = 219733500$$

$$225523743 \div 230270081 = \cdot 97938795$$

$\therefore$ The common logarithm of 95364768 = 7.97938795.

This result may be found without the use of common division, thus:—

$$23 \times 98 = 2254$$

$$2254 \downarrow 0,0,0,5,5,\bar{1},\bar{1},\bar{2}, = 225523743$$

$$\therefore \frac{98 \times 23 \downarrow 0,0,0,5,5,\bar{1},\bar{1},\bar{3},}{23 \downarrow 0,0,1,1,7,4,0,8,} = 98 \downarrow 0,0,\bar{1},4,\bar{2},\bar{5},\bar{1},\bar{1},$$

$$\cdot 9793879598 = 98 \downarrow 0,0,\bar{1},3,7,4,7,9,$$

The log. of this constant number is wrong in Baron Von Vega's Tables, by Fischer, published at Berlin, 1857.

6. *How many degrees, minutes, &c. are contained in an arc of a circle, length = .34567895, radius = 1?*

RULE.

Multiply double the length of the given arc by 100000, and then by $\downarrow 0,3,1,0,0,\bar{7},0,\bar{5},$

$$\begin{array}{r|l} 69 & 1\ 3\ 5\ 7\ 90 \\ 20 & 7\ 4\ 0\ 74 \\ & 2\ 0\ 7\ 41 \\ & 69 \end{array} \quad 0,3,1,0,0,7,0,5,$$

$$\begin{array}{r|l} 71\ 2 & 3\ 0\ 6\ 74 \\ & 7\ 1\ 2\ 31 \end{array}$$

$$\begin{array}{r|l} 71\ 3\ 0\ 1 & 9\ 05\ \dots \\ & 4\ 99\ \dots \\ & 4\ \dots \end{array} \left. \vphantom{\begin{array}{r|l} 71\ 3\ 0\ 1 \\ 4\ 99 \\ 4 \end{array}} \right\} \text{minus}$$

$$19^\circ\ 48'\ 21''\cdot402 = 71\ 3\ 0\ 1''\cdot402$$

7. What is the length of an arc that contains $19^\circ\ 48'\ 21''\cdot402 = 71301''\cdot402$, radius = 1?

RULE.

Subtract twice the number of seconds from half the number of seconds with two cyphers affixed; the remainder, multiplied by $\downarrow 0,1,0,0,2,8,2,2$, gives 100000 times the length of the given arc.

$$\begin{array}{r} 2\)\ 71\ 30\ 1''\cdot40200 \\ \hline 35\ 65\ 0\ 70\ 100 \\ 1\ 42\ 6\ 02\ 804 \\ \hline 34\ 22\ 4\ 67\ 296 \\ 34\ 2\ 24\ 673 \\ \hline 34\ 56\ 6\ 91\ 969 \\ 6\ 91\ 34\ \dots \\ 2\ 7\ 6\ 54\ \dots \\ 6\ 91\ \dots \\ 69\ \dots \end{array} \quad \begin{array}{l} \text{Constant.} \\ \downarrow 0,1,0,0,2,8,2,2, \end{array}$$

$$\text{Length of arc} = \underline{34\ 56\ 7\ 89\ 497}$$

8. Given the obliquity of the Elliptic = $23^\circ\ 27'\ 25''\cdot42 = 84445''\cdot42$, to find the natural sine, and the log. sin of this angle.

$$\begin{array}{r} 2\)\ 84445''\cdot4200 \\ \hline 42222\ 7100 \\ 1688\ 9084 \\ \hline \end{array} \quad \text{Constant.}$$

$$\text{Length of arc} = 40533\ 8016\ \downarrow 0,1,0,0,2,8,2,2, = 409402949$$

$$\cdot 409402949 = \cdot 4 \downarrow 0,2,3,3,3,6,1,8, = \cdot 4 \downarrow \overline{2323649}, |.$$

This last result may be found by a single operation, since the constant $\downarrow 0,1,0,0,2,8,2,2,$ is given.

$$\begin{aligned} \text{Constant} &= \cdot 4 \downarrow 0,1,3,3,0,7,9,6, = \cdot 405338016 \\ &\quad \downarrow 0,1,0,0,2,8,2,2, \\ &\quad \cdot 4 \downarrow 0,2,3,3,3,6,1,8, \text{ Length of arc, which} \\ \text{put} &= x. \text{ It is well-known that} \end{aligned}$$

$$\sin x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \dots$$

$$\begin{array}{r} 4^{\cdot} = 138636402 \\ \quad 2323649 \\ \hline 140960051 = 4 \downarrow \overline{2323649}, | \\ \quad 3 \\ \hline \text{cube} \dots 422880153 \\ \quad 179184951 = 2.3 \\ \hline \quad 243695202 \\ 11^{\cdot} \dots 239801578 \\ \hline \quad 3893624 \\ \downarrow 0,3, \dots 2985249 = 3B \\ \hline \quad 908375 \\ \downarrow 0,0,9, \dots 899595 = 9C \\ \hline \quad 0,8,7,8,0, \end{array}$$

Dividing by $(10)^8$; $11 \downarrow 0,3,9,0,8,7,8,0, = \cdot 011436724$ minus

$$\begin{array}{r} \quad \quad \quad 140960051 \\ \quad \quad \quad 5 \\ \hline \text{Fifth power} \dots 704800255 \\ \quad 478773232 = 2.3.4.5 \\ \hline \quad 226027023 \\ 9^{\cdot} = 219733500 \\ \hline \quad 6293523 \end{array}$$

$$\begin{array}{r}
 \downarrow 0,6, \dots \quad 629\,3\,5\,2\,3 \\
 \quad 597\,0\,4\,9\,8 = 6B \\
 \hline
 \downarrow 0,0,3, \dots \quad 32\,3\,0\,2\,5 \\
 \quad 29\,9\,8\,6\,5 = 3C \\
 \hline
 \quad 2,3,1,6,0,
 \end{array}$$

$\therefore$ Dividing by $(10)^5$; $9 \downarrow 0,6,3,2,3,1,6,0 = \cdot 000095846$ plus.

$$\begin{array}{r}
 \quad 14096\,0\,0\,5\,1 \\
 \quad \quad 7 \\
 \hline
 \quad 98672\,0\,3\,5\,7 \\
 \quad 85255\,8\,9\,7\,8 = 2.3.4.5.6.7 \\
 \hline
 3 \cdot \dots \quad 13416\,1\,3\,7\,9 \\
 \quad 10986\,6\,7\,5\,0 \\
 \hline
 \quad 2429\,4\,6\,2\,9 \\
 \downarrow 2, \dots \quad 1906\,2\,9\,9\,4 = 2A \\
 \hline
 \quad 523\,1\,6\,3\,3 \\
 \downarrow 0,5, \dots \quad 497\,5\,4\,1\,5 = 5B \\
 \hline
 \quad 25\,6\,2\,2\,0 \\
 \downarrow 0,0,2, \dots \quad 1999\,1\,0 = 2C \\
 \hline
 \quad 5,6,3,1,0,
 \end{array}$$

Dividing by $(10)^7$; $3 \downarrow 2,5,2,5,6,3,1 = 000000382$ minus.

$$\begin{array}{r}
 \cdot 409402949 + \\
 1\,1436724 - \\
 95846 + \\
 382 - \\
 \hline
 \end{array}$$

$$\text{Nat. sin } 23^\circ 27' 25'' \cdot 42 = \underline{\underline{\cdot 398061689}}$$

$$3 \overline{) 3'98061689}$$

$$1'32687229 = \downarrow 2,9,2,6,5,2,2,1, = \downarrow \overline{28283872,}$$

$$3' = \downarrow \overline{109866750,}$$

$$\therefore 3'98061686 = \downarrow \overline{138150622,}$$

Then 138150622, divided by the constant 230270081, gives
 59995038.

$$\therefore \text{Log. sin } 23^\circ 27' 25'' \cdot 42 = 9 \cdot 59995038.$$

9. Find the degrees minutes, seconds, &c. in an arc to radius 1,
 whose sine = $\cdot 226941796 = 2 \downarrow 1,3,1,2,1,5,5,5, = \overline{81956457,}$

$$\begin{array}{r}
 \phantom{\text{cube}} 8195\,6457 \\
 \phantom{\text{cube}} 3 \\
 \hline
 \text{cube } \dots\dots\dots 24586\,9371 \\
 \phantom{\text{cube}} 17918\,4951 = \frac{1.}{2.3} \\
 \hline
 \phantom{\text{cube}} 6668\,4420 \\
 \downarrow 6, \dots\dots 5718\,8982 = 6\,A \\
 \hline
 \phantom{\text{cube}} 949\,5438 \\
 \downarrow 0,9, \dots\dots 895\,5747 = 9\,B \\
 \hline
 \phantom{\text{cube}} 53\,9691 \\
 \downarrow 0,0,5, \dots 499\,775 = 5\,C \\
 \hline
 \phantom{\text{cube}} 3,9,9,1,6,
 \end{array}$$

Dividing by $(10)^3 \downarrow 6,9,5,3,9,9,1,6$, becomes $\cdot 001948014$

$$\begin{array}{r}
 \phantom{\text{fifth power}} 8195\,6457 \\
 \phantom{\text{fifth power}} 5 \\
 \hline
 \text{fifth power } \dots\dots\dots 40978\,2285 \\
 \phantom{\text{fifth power}} 25903\,9732 = \frac{1.3}{2.4.5} \\
 \hline
 \phantom{\text{fifth power}} 15074\,2553 \\
 4' \dots\dots\dots 13863\,6402 \\
 \hline
 \phantom{\text{fifth power}} 12106\,151 \\
 \downarrow 1, 953\,1497 = A \\
 \hline
 \phantom{\text{fifth power}} 257\,4654 \\
 \downarrow 0,2, \dots\dots 199\,0166 = 2\,B \\
 \hline
 \phantom{\text{fifth power}} 58\,4488 \\
 \downarrow 0,0,5, \dots\dots 499\,775 = 5\,C \\
 \hline
 \phantom{\text{fifth power}} 8,4,7,1,3,
 \end{array}$$

Dividing $(10)^5 4 \downarrow 1,2,5,8,4,7,1,3$, gives $\cdot 000045146$.

$$\begin{array}{r}
 8195\ 645\ 7 \\
 \hline
 573695\ 199 \\
 31092\ 1720 = \frac{1.3.5}{2.4.6.7} \\
 \hline
 26277\ 3479 \\
 10 \cdot \dots\dots 23027\ 0081 \\
 \hline
 \downarrow 3, \dots\dots 32503\ 398 \\
 28594\ 491 = 3\ A \\
 \hline
 \downarrow 0, 3, \dots\dots 39089\ 07 \\
 2985\ 249 = 3\ B \\
 \hline
 92365\ 8 \\
 89959\ 5 = 9\ C \\
 \hline
 2,4,0,6,3,
 \end{array}$$

Dividing by $(10)^7$ $10 \downarrow 3,3,9,2,4,0,6,3$, becomes $\cdot 000001384$.

$$\begin{array}{r}
 \text{sine } \cdot 22694\ 1796 \\
 \cdot 00194\ 8014 \\
 \cdot 00004\ 5146 \\
 \cdot 00000\ 1384 \\
 \hline
 \text{arc } \cdot 22893\ 6340 \\
 2
 \end{array}$$

$$\text{seconds in arc} = 45787 \cdot 2680 \downarrow 0,3,1,0,0,\overline{7},0,\overline{5},$$

$$45787 \cdot 27 \downarrow 0,3,1,0,0,\overline{7}, \text{ will give the seconds sufficiently correct,} \\ = 47221'' \cdot 42 = 13^\circ 7' 1'' \cdot 42.$$

10. Find the area of the curve whose equation is

$$y = \frac{2}{\sqrt{\pi}} \varepsilon^{-x^2}$$

$$\pi = 3 \downarrow 0,4,6,3,1,9,3,0, = \overline{114478742,}$$

$$\varepsilon = \downarrow 10,4,7,1,0,0,3,8, = \overline{100005025,}$$

$$10\ y = 4 \cdot 2087962 = 4 \downarrow 0,5,1,1,3,1,2,0, = \downarrow \overline{143724892,}$$

or $y = \cdot 42087962$. being given.

$$\epsilon^{x^2} = \frac{2}{y\sqrt{\pi}} = \frac{20}{10y\sqrt{\pi}}$$

$$10 = 230270081$$

$$2 = 69318201$$

$$299588282$$

$$57239371 \dots \sqrt{\pi}$$

$$242348911$$

$$10y = 143724892$$

$$98624019$$

When required, x is readily found; for 98624019 divided by 100005025, gives x^2 . $\therefore x = .99307$.

$$x^2 = \frac{98624019}{100005025} = \frac{9 \downarrow 0,9,2,0,4,7,0,0,}{10 \downarrow 0,0,0,0,5,0,2,5,} = \frac{\downarrow 228884457,}{\downarrow 230275106,}$$

$$= -1390649,$$

$$\therefore x = -695325,$$

The area between the limits $x = 0$ and $x = x$, may be expressed by

$$\frac{2}{\sqrt{\pi}} \left(x - \frac{1}{1 \cdot 3} x^3 + \frac{1}{1 \cdot 2 \cdot 5} x^5 - \frac{1}{1 \cdot 2 \cdot 3 \cdot 7} x^7 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 9} x^9 - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 11} x^{11} + \dots \right)$$

$$2 \dots \dots \dots + 69318201$$

$$- 57239371 \dots \sqrt{\pi}$$

$$\frac{2}{\sqrt{\pi}} \dots \dots \dots 12078830$$

$$- 695325 \dots x$$

$$\frac{2}{\sqrt{\pi}} x \dots \dots + 11383505 = \downarrow 1,1,8,5,7,2,8,5, = 1.119983 +$$

$$9531497$$

$$1852008$$

$$995083$$

$$856925$$

$$799640$$

$$5,7,2,8,5,$$

$$\begin{array}{r} 3 \dots\dots\dots - 109866750 \\ x^3 \dots\dots - 2085975 \end{array}$$

$$\begin{array}{r} - 111952725 \\ \frac{2}{\sqrt{\pi}} \dots\dots + 12078830 \end{array}$$

$$\begin{array}{r} - 99873895 \\ 10 \dots\dots + 230270081 \end{array}$$

$$130396186 = 3 \downarrow 2, 1, 4, 7, 1, 5, 3, 9, \dots \cdot 368362 -$$

$$5 \dots\dots - 160951879$$

$$2 \dots\dots - 69318201$$

$$x^5 \dots\dots - 3476625$$

$$ - 233746705$$

$$+ 12078830 \dots \frac{2}{\sqrt{\pi}}$$

$$ - 221667875$$

$$10 \dots\dots + 230270081$$

$$8602206 = \downarrow 0, 8, 6, 6, 1, 8, 1, 2, \dots \cdot 1090044 +$$

$$7 \dots\dots - 194600795$$

$$1.2.3 - 179184951$$

$$x^7 \dots\dots - 4867275$$

$$ - 378653021$$

$$+ 12078830 \dots \frac{2}{\sqrt{\pi}}$$

$$ - 366574191$$

$$(10)^2 \dots\dots + 460540162$$

$$ + 93965971$$

$$2 \dots\dots - 69318201$$

$$+ 24647770 = \downarrow 2, 5, 6, 0, 9, 6, 3, 1, \dots \cdot 025590 -$$

$$9 \dots\dots - 219733500$$

$$1.2.3.4 \dots\dots - 317821353$$

$$x^9 \dots\dots - 6257925$$

$$ - 543812778$$

$$+ 12078830 \dots \frac{2}{\sqrt{\pi}}$$

$$ - 531733948$$

$$(10)^8 \dots\dots + 690810243$$

$$159076295$$

$$\begin{array}{r}
 159076295 = 4 \downarrow 2,1,3,8,1,9,5,1, \\
 4 \dots\dots\dots 138636402 \\
 \hline
 20439893
 \end{array}$$

Divide by $(10)^8$, the result will be $+ \cdot 004907$, the next term.

$$\begin{array}{r}
 11 \dots\dots\dots - 239801578 \\
 1.2.3.4.5 \dots - 478773232 \\
 x^{11} \dots\dots - 7648575 \\
 \hline
 - 726223385 \\
 + 12078830 \dots\dots\dots \frac{2}{\sqrt{\pi}} \\
 \hline
 - 714144555 \\
 10^4 \dots + 922080324 \\
 \hline
 + 207935769 = 8 \downarrow 0,0,0,1,8,8,3,5, = \cdot 000800 \\
 8 \dots - 207954604 \\
 \hline
 \text{negative} \quad 1,8,8,3,5,
 \end{array}$$

7.9986 divided by 10^4 gives $\cdot 000800 -$

The next step gives $\cdot 000111 +$

$$\begin{array}{r}
 1.119983 + \\
 368362 - \\
 \hline
 \cdot 751621 \\
 \cdot 109004 + \\
 \hline
 \cdot 860625 \\
 025590 - \\
 \hline
 \cdot 834035 \\
 \cdot 004907 + \\
 \hline
 \cdot 838942 \\
 \cdot 000800 - \\
 \hline
 \cdot 838142 \\
 \cdot 000111 + \\
 \hline
 \text{Area of curve} = \cdot 838253
 \end{array}$$

11. Find the ordinate y , and area of the curve whose equation is

$$y = \frac{2}{\sqrt{\pi}} \epsilon^{-x^2} = \frac{2}{\sqrt{\pi} \epsilon^{x^2}}$$

between the limits $x = 0$, and $x = 2.12084$.

$$x^2 = 4.497962 = 4 \downarrow 1,2,2,1,2,0,3,0, = \downarrow 150370005,$$

$$2 \dots\dots + 69318201$$

$$\sqrt{\pi} \dots\dots - 57239371$$

$$+ 12078830$$

$$\epsilon^{x^2} \dots\dots - 449818802 = 4.497962 \times 100003025$$

$$- 437739972 = \downarrow \bar{y},$$

$$10^2 \dots + 460540162$$

$$22800190 = \downarrow 2,3,7,5,2,1,6,2, = 1.25607209$$

$$\therefore y = .0125607209$$

The area of this curve between the limits $x = 0$, and $x = x$ may be expressed by

$$1 - \frac{1}{x \sqrt{\pi} \epsilon^{x^2}} \left(1 - \frac{1}{2x^2} + \frac{1.3}{2^2 x^4} - \frac{1.3.5}{2^3 x^6} + \dots \right)$$

$$\frac{1}{x \sqrt{\pi} \epsilon^{x^2}} = - 582243176$$

$$10^3 = + 690810243$$

$$108567067 = 2 \downarrow 4,1,1,2,7,8,4,0, = 2.9612632$$

$$\therefore \text{First term} = .0029613$$

$$\frac{1}{2x^3 \sqrt{\pi} \epsilon^{x^2}} = - 801931382$$

$$+ 921080324 = 10^4$$

$$119148942 = 3 \downarrow 0,9,3,2,6,5,8, = 3.2917839$$

$$\therefore \text{Second term} = .0003292.$$

$$\frac{1.3}{2^2 x^5 \sqrt{\pi} \epsilon^{x^2}} = - \frac{911752838}{921080324} = 10^4$$

$$9327486 = \downarrow 0,9,3,7,1,8,7,3, = 1.0977584$$

$\therefore$ Third term = .0001098.

$$\frac{1.3.5}{2^3 x^7 \sqrt{\pi} \epsilon^{x^2}} = - \frac{970489165}{1151350405} = 10^5$$

$$180861240 = 6 \downarrow 0,1,6,8,1,4,7,6,$$

$$= (1.0169034) \times 6$$

$\therefore$ Fourth term = .0000610.

The fifth term will be = .0000474577; hence

$$\begin{array}{r} 1.0000000 + \\ .0029613 - \\ \hline .9970387 \\ 3292 + \\ \hline .9973679 \\ 1098 - \\ \hline .9972581 \\ 610 + \\ \hline \text{Area} = .9973191 \text{ nearly.} \end{array}$$

12. The apparent distance of the centres of the planet Venus and the moon = $76^\circ 14' 47''$ (D), the apparent altitude of the moon's centre = $46^\circ 36' 27''$ (A), the true altitude = $47^\circ 12' 47''$ (a), the apparent altitude of Venus = $20^\circ 25' 10''$ (B), and her true altitude = $20^\circ 22' 54''$ (b); determine the true distance (d), so that the longitude of the place of observation may be found.

$$\cos d = \left\{ \cos D + \cos (A + B) \right\} \frac{\cos a \cos b}{\cos A \cos B} - \cos (a + b).$$

$$\begin{aligned}\cos (A + B) &= \cdot 3902982 & \cos (a + b) &= \cdot 3811556 \\ \cos D &= \cdot 2377471 \\ &\underline{\cdot 6280453}\end{aligned}$$

$$\begin{aligned}\cos a &= \cdot 6792741 & \cos b &= \cdot 9373934 \\ \cos A &= \cdot 6869924 & \cos B &= \cdot 9371636\end{aligned}$$

$$\cos a \downarrow 0,1,1,3,4,8,7, = \cos A$$

$$\cos B \downarrow 0\ 0\ 0,2,4,5,2, = \cos b$$

$$\therefore \cos d = \cdot 6280453 \frac{\downarrow 0,0,0,2,4,5,2,}{\downarrow 0,1,1,3,4,8,7,} - \cdot 3811556$$

$$= \cdot 6280453 \downarrow 0,1,1,1,0,3,5, - \cdot 3811556$$

$$= \cdot 2399859 \cos \text{ of } 76^\circ 6' 51'' \cdot 4, \text{ the true distance.}$$

13. Find the value of x in the equation

$$347 \cdot 6392 x^3 - 84 \cdot 35216 x^2 + 413 \cdot 6645 x = 4582575 \cdot 36; (R).$$

Omitting the decimals, and examining the equation

$$347 x^3 - 84 x^2 + 413 x = 4500000,$$

it is easily observed that a value of x lies between 10 and 30. If 20 be substituted for x , the result will be

$$+ 2781113 \cdot 6 - 33740 \cdot 864 + 8273 \cdot 29 = 2755646 \cdot 026; (r).$$

The result (r) is sufficiently near to (R) to effect the solution, for it is possible to render $r \downarrow a_1, a_2, a_3, a_4, \dots = R$.

278	once	+ 2 78
2	$\frac{2}{3}$	- 3
0	$\frac{1}{3}$	+
276		2 75 take
		4 58 from (R)
		2 76) 1 83 (\downarrow 6,

The quotient $\downarrow 6$, may be found by mere observation, without setting down any figures, and the process may be continued by operating with $\downarrow 5$, for x^3 .

$\downarrow 5,$ $+ 278111360$ $+ 447901127$	$\downarrow 3,3,1,9,1,9,6,2,$ $- 3374086$ $- 4635874$	$\downarrow 1,6,3,8,3,9,6,8,$ $+ 827329$ $+ 969763$
---	---	---

$$\begin{array}{r} 44 \overline{) 79} + \\ \quad 30 - \\ \quad \quad 3 + \\ \hline 44 \overline{) 52} \end{array}$$

once
 $\frac{2}{3}$ of
 $\frac{1}{3}$ of

$$\begin{array}{r} + 44 \overline{) 79} \dots\dots \\ - \quad 46 \dots\dots \\ + \quad \quad 9 \dots\dots \\ \hline 44 \overline{) 42} \dots\dots \text{take} \\ 45 \overline{) 82} \dots\dots \text{from } (R) \\ \hline 1 \overline{) 40} (\downarrow 0,3, \end{array}$$

$$\begin{array}{r} \downarrow 0,3, \\ + 447901127 \\ + 461472979 \end{array}$$

$$\begin{array}{r} \downarrow 0,2, \\ - 4635874 \\ - 4729065 \end{array}$$

$$\begin{array}{r} \downarrow 0,1, \\ + 969763 \\ + 979461 \end{array}$$

$$\begin{array}{r} 46 \overline{) 4} + \\ \quad 2 - \\ \quad \quad 3 + \\ \hline 458 \overline{) 5} \end{array}$$

once
 $\frac{2}{3}$ of
 $\frac{1}{3}$ of

$$\begin{array}{r} + 461 \overline{) 47} \dots\dots \\ - \quad 4 \overline{) 72} \dots\dots \\ + \quad \quad 97 \dots\dots \\ \hline 457 \overline{) 72} \dots\dots \text{take} \\ 458 \overline{) 25} \dots\dots \text{from } (R). \\ \hline 153 \dots\dots (\downarrow 0,0,1, \end{array}$$

$$\begin{array}{r} \downarrow 0,0,1, \\ + 461472979 \\ + 461934452 \end{array}$$

$$\begin{array}{r} \downarrow 0,0,0,6,6,6,3,7, \\ - 4729065 \\ - 4732216 \end{array}$$

$$\begin{array}{r} \downarrow 0,0,0,3,3,3,1,8, \\ + 979461 \\ + 979787 \end{array}$$

$$\begin{array}{r} + 4619 \overline{) 34} \dots\dots \\ - \quad 47 \overline{) 32} \dots\dots \\ + \quad \quad 9 \overline{) 79} \dots\dots \\ \hline 4581 \overline{) 82} \dots\dots \text{take} \\ 4582 \overline{) 57} \dots\dots \text{from } (R) \\ \hline 175 (\downarrow 0,0,0,1, \end{array}$$

$$\begin{array}{rcl}
 \downarrow 0,0,0,1, & \downarrow 0,0,0,0,6,6,6,6, & \downarrow 0,0,0,0,3,3,3,3 \\
 + 461934452 & - 4732216 & + 979787 \\
 + 461980645 & - 4732531 & + 979819
 \end{array}$$

$$\begin{array}{rcl}
 46198 + & \text{once} & + 46198|0645 \\
 315 - & \frac{2}{3} \text{ of} & - 473|2531 \\
 \hline
 32 + & \frac{1}{3} \text{ of} & + 45724|8114 \\
 45915) & & 97|9819 \\
 \dots & & \hline
 & & 45822|7933 \text{ take} \\
 & & 45825|7536 \text{ from } (R) \\
 & & \hline
 & & 2|9603 \\
 & & 2|7549 \text{ } (\downarrow 0,0,0,0,6,4,4,7, \\
 & & \hline
 & & 2054 \\
 & & 1837 \\
 & & \hline
 & & 217 \\
 & & 184 \\
 & & \hline
 & & 33 \\
 & & 32 \\
 & & \hline
 \end{array}$$

$$\therefore x^3 = (20)^3 \downarrow 5,3,1,1,6,4,4,7,$$

$$\therefore x = (20) \downarrow 1,7,4,2,2,7,6,9, = 23.6868595 (R).$$

14. *Given, $x^2 + 1.41421356 x = 1.73205081$, to find the value of x .*

As x is situated between $\cdot 1$ and 1 , operations may be commenced with either $\cdot 5$, $\cdot 6$, or $\cdot 7$. When $x = \cdot 7$, the equation becomes

$$+ 4900000000 + 989949492 = 1.479949492$$

$$\begin{array}{rcl}
 9|80 + & \text{twice} & + 4|90 \dots\dots\dots \\
 9|89 + & \text{once} & + 9|89 \dots\dots\dots \\
 \hline
 19|69 & & \hline
 & & 14|79 \dots\dots\dots \text{ take} \\
 & & 17|32 \dots\dots\dots \text{ from } (R) \\
 & & \hline
 & & 2|53 \text{ } (\downarrow 1,
 \end{array}$$

$$\begin{array}{r} \downarrow 2, \\ + 490000000 \\ + 592900000 \end{array}$$

$$\begin{array}{r} 11|858 + \text{twice} \\ 10|889 + \text{once} \\ \hline 22|747 \end{array}$$

$$\begin{array}{r} \downarrow 1, \\ + 989949492 \\ + 1088944441 \end{array}$$

$$\begin{array}{r} + 5|929 \dots\dots \\ + 10|889 \dots\dots \\ \hline 16|818 \dots\dots \text{take} \\ 17|320 \dots\dots \text{from } (R) \\ \hline 1502 (\downarrow 0,2, \end{array}$$

$$\begin{array}{r} \downarrow 0,4, \\ + 592900000 \\ + 616974118 \end{array}$$

$$\begin{array}{r} 123|38 + \text{twice} \\ 111|08 + \text{once} \\ \hline 234|46 \end{array}$$

$$\begin{array}{r} \downarrow 0,2, \\ + 1088944441 \\ + 1110832224 \end{array}$$

$$\begin{array}{r} + 61|69 \dots\dots \\ + 111|08 \dots\dots \\ \hline 172|77 \dots\dots \text{take} \\ 173|20 \dots\dots \text{from } (R) \\ \hline 43 (\downarrow 0,0,1, \end{array}$$

$$\begin{array}{r} \downarrow 0,0,2, \\ + 616974118 \\ + 618208683 \end{array}$$

$$\begin{array}{r} \downarrow 0,0,1, \\ + 1110832224 \\ + 1111943056 \end{array}$$

$$\begin{array}{r} 12364 + \text{twice} \\ 11119 + \text{once} \\ \hline 23483 \end{array}$$

$$\begin{array}{r} + 618|208683 \\ + 1111|943056 \\ \hline 1730|151739 \text{ take} \\ 1732|050810 \text{ from } (R) \\ \hline \end{array}$$

$$\begin{array}{r} 1|899071 \\ 1|87864 (\downarrow 0,0,0,8,0,8,6, \\ \hline 2043 \\ 1879 \\ \hline 164 \end{array}$$

$$\therefore x = \cdot 7 \downarrow 1,2,1,8,0,8,6, = \cdot 7868982.$$

d

15. Find a value of x in the equation

$$2.7634x^5 + 12.349x^4 - 542.36x^3 - 7621.3x = -174859.34 (R).$$

A value of x evidently lies between 1 and 10; 5, 6, or 7, may be substituted for x , but 7 is most convenient. 7 being substituted, the equation becomes

$$+46444.4638 + 29649.949 - 186029.48 - 53349.7 = -163284.13872.$$

Working to three places of decimals to find x^5 , the first factor may be found thus:—

$$\begin{array}{r} + 4 \overline{)64} + \\ + 2 \overline{)37} + \\ - 11 \overline{)16} - \\ - 1 \overline{)06} - \\ \hline 5 \overline{)21} \end{array}$$

once
 $\frac{4}{5}$ of
 $\frac{3}{5}$ of
 $\frac{1}{5}$ of

$$\begin{array}{r} + 4 \overline{)64} \dots \dots \\ + 2 \overline{)96} \dots \dots \\ - 18 \overline{)60} \dots \dots \\ - 5 \overline{)33} \dots \dots \\ \hline - 16 \overline{)32} \dots \dots \text{take} \\ - 17 \overline{)48} \dots \dots \text{from } (R) \\ \hline - 116 (\downarrow 2, \end{array}$$

$$\begin{array}{cccc} \downarrow 2, & \downarrow 1,5,7,4,3,7,9,8, & \downarrow 1,1,9,1,1,6,2,1, & \downarrow 0,3,8,2,7,7,1, \\ + 46444.464 & + 29649.949 & - 186029.480 & - 53349.100 \\ + 56197.802 & + 34534.427 & - 208570.554 & - 55422.253 \end{array}$$

$$\begin{array}{r} + 56 \overline{)1} + \\ + 27 \overline{)6} + \\ - 125 \overline{)1} - \\ - 11 \overline{)0} - \\ \hline - 52 \overline{)4} \end{array}$$

once
 $\frac{4}{5}$ of
 $\frac{3}{5}$ of
 $\frac{1}{5}$ of

$$\begin{array}{r} + 56 \overline{)197} \dots \dots \\ + 34 \overline{)534} \dots \dots \\ - 208 \overline{)570} \dots \dots \\ - 55 \overline{)422} \dots \dots \\ \hline - 173 \overline{)260} \dots \dots \text{take} \\ - 174 \overline{)859} \dots \dots \text{from } (R) \\ \hline - 11598 \dots \dots (\downarrow 0,3, \end{array}$$

$$\begin{array}{cccc} \downarrow 0,3, & \downarrow 0,2,3,9,8,1,6,8, & \downarrow 0,1,7,9,6,3,8,1, & \downarrow 0,0,5,9,7,2,7,5, \\ + 56197.802 & + 34534.427 & - 208570.554 & - 55422.253 \\ + 57900.651 & + 35369.063 & - 212339.831 & - 55754.126 \end{array}$$

$$\begin{array}{r|l}
 + 579 & \text{once} \\
 + 282 & \frac{4}{5} \text{ of} \\
 - 1274 & \frac{3}{5} \text{ of} \\
 - 111 & \frac{1}{5} \text{ of} \\
 \hline
 - 523 &
 \end{array}$$

$$\begin{array}{r|l}
 + 579 & 00 \dots \\
 + 353 & 69 \dots \\
 - 2123 & 39 \dots \\
 - 557 & 54 \dots \\
 \hline
 1748 & 24 \dots \\
 1748 & 59 \dots \\
 \hline
 &
 \end{array}$$

$$|35 \dots (\downarrow 0,0,0,6,$$

$$\downarrow 0,0,0,6,$$

$$+ 57900651$$

$$+ 57935400$$

$$\downarrow 0,0,0,4,8,$$

$$+ 35369063$$

$$+ 35386044$$

$$\downarrow 0,0,0,3,6,$$

$$- 212339831$$

$$- 212416283$$

$$\downarrow 0,0,0,1,2,$$

$$- 55754126$$

$$- 55760816$$

$$\begin{array}{r|l}
 + 57935 & \text{once} \\
 + 28309 & \frac{4}{5} \text{ of} \\
 \hline
 + 86244 & \\
 - 127449 & \frac{3}{5} \text{ of} \\
 \hline
 - 41205 & \\
 - 11152 & \frac{1}{5} \text{ of} \\
 \hline
 - 52357 & \\
 + &
 \end{array}$$

$$\begin{array}{r|l}
 + 57935 & 400 \\
 + 35386 & 044 \\
 \hline
 + 93321 & 444 \\
 - 212416 & 283 \\
 \hline
 - 119094 & 839 \\
 - 55760 & 816 \\
 \hline
 - 174855 & 655 \text{ take} \\
 - 174859 & 340 \text{ from } (R) \\
 \hline
 3 & 685 \\
 3 & 665 (\downarrow 0,0,0,0,7,
 \end{array}$$

$$\therefore x^5 = 7^5 \downarrow 2,3,0,6,7,0,0,0,$$

$$\therefore x = 7 \downarrow 0,4,4,4,2,8,9,6, = 7.31654917$$

16. Find the value of x to eight places of figures in the equation

$$\begin{aligned}
 789x^7 - 678x^6 + 567x^5 + 456x^4 - 345x^3 - 234x^2 + 123x \\
 = 965432101234567. \quad (R).
 \end{aligned}$$

A value of x lies between 0, and 100; take $x = 50$, then the result will be

$$\begin{aligned}
 &616406250000000 - 10593750000000 + 177187500000 \\
 &+ 2850000000 - 43125000 - 585000 + 6150 \\
 &= 605992493796150.
 \end{aligned}$$

Since the value of x is only required to eight places of figures, the numbers to be operated upon will be

$\downarrow 4,$ + 616406250	$\downarrow 3,4,1,0,4,6,4,$ - 10593750	$\downarrow 2,8,1,0,9,3,$ + 177188	$\downarrow 2,2,7,3,$ + 2850
	$\downarrow 1,7,$ - 43	- 1	= 605992494
$\downarrow 0,8,$ + 902480319	$\downarrow 0,6,8,5,3,2,8,8,$ - 14688162	$\downarrow 0,5,7,1,1,$ + 232416	$\downarrow 0,4,5,7,$ + 3544
	$\downarrow 0,3,$ - 50	- 1	= 888028066
$\downarrow 0,0,3,$ + 977256943	$\downarrow 0,0,2,5,7,1,1,7,$ - 15725329	$\downarrow 0,0,2,1,4,3,$ + 246014	$\downarrow 0,0,1,7,$ + 3709
	- 52	- 1	= 961781284
$\downarrow 0,0,0,7,$ + 980191647	$\downarrow 0,0,0,6,$ - 15765797	$\downarrow 0,0,5,$ + 246542	$\downarrow 0,0,0,4,$ + 3716
	- 52	- 1	= 964676055
+ 980877877	- 15775258	+ 246665	+ 3717
	- 52	- 1	= 965352948

The divisor that determines the next operating numbers, may be employed to find the remaining figures that compose the root; the work will stand as follows:

+ 98088	once	+ 98087 7877
- 1352	$\frac{6}{7}$ of	- 1577 5258
96736		96510 2619
18	$\frac{5}{7}$ of	+ 24 6665
0	$\frac{4}{7}$ of	+ 3717
96754		96535 3001
...†		53
		96535 2948

$$\begin{array}{r} 96535 | 2948 \text{ take} \\ 96543 | 2101 \text{ from } (R) \\ \hline \end{array}$$

$$\begin{array}{r} 7 | 9153 \\ 7 | 7403 \text{ (8,} \\ \hline \end{array}$$

$$\begin{array}{r} | 1750 \\ 968 \text{ (1,} \\ \hline \end{array}$$

$$\begin{array}{r} 782 \\ 774 \text{ (8,} \\ \hline \end{array}$$

$$\begin{array}{r} 8 \\ 9 \text{ (1,} \\ \hline \end{array}$$

$$x^7 = (50)^7 \downarrow 4,8,3,7,8,1,8,1,$$

$$\therefore x = 50 \downarrow 0,6,6,6,7,5,8,6,$$

$$\therefore x = 53.4313588$$

17. Given $x_x = 72.69517$, to find the value of x .

$$72.69517 = 10.7 \downarrow 0,3,7,9,3,2,2,6, = \downarrow \overline{428649036,}$$

Because $3 = \downarrow \overline{109866750,}$ and $4 = \downarrow \overline{138636402,}$ the value of x must lie between 3 and 4; for if $\downarrow \overline{X,}$ represents the reduction of x , in the same way that $\downarrow \overline{109866750,}$ represents 3, then

$$x \times X = 428649036.$$

Consequently 3×109866750 is too small
and 4×138636402 is too great

$\therefore$ The first part of the expression for x may be represented by $3 \downarrow 1$, Putting A for 9531497, B for 995083, and C for 99955, D for 10000, &c., the value of x may be found as follows:—

$$\downarrow \overline{428649036,} (n)$$

$$\begin{array}{rcl}
 & 3 = 109866750 & 995083(B) \\
 \downarrow 1 = & 9531497(A) & \underline{3} \\
 & 119398247 & 2 \overline{) 985249} \\
 & \underline{3} & \underline{298525} \\
 & 3 \overline{) 58194741} & 3283774(b) \\
 & \underline{35819474} & \\
 (o) 394... (n) 4286... & & 99955(c) \\
 (b) 328... (o) 3940... & & \underline{3} \\
 722) & 346(\downarrow 0,4, & 2 \overline{) 99865} \\
 & & \underline{29987} \\
 & & 329852 \\
 & & 13194 \\
 & & 198 \\
 & & \underline{1} \\
 & & 343245(C) \\
 (p) 4236... (n) 42864... & & 10000(D) \\
 (c) 3432... (p) 42368... & & \underline{3} \\
 7668) & 496(\downarrow 0,0,6, & 3 \overline{) 0000} \\
 & & \underline{3000} \\
 & & 33000. \\
 & & 1320. \\
 & & 20. \\
 & & \underline{34340.} \\
 & & 206. \\
 & & \underline{1.} \\
 & & 34547 \\
 (q) 428301... (n) 428649... & & 1000(E) \\
 (d) 34547 (q) 428301... & & \underline{3} \\
 773771) & 348(\downarrow 0,0,0,4, & 3 \overline{) 000} \\
 & & \underline{300} \\
 & & 3300 \\
 & & 132 \\
 & & \underline{2} \\
 & & 3434.. \\
 & & 21.. \\
 & & \underline{3455}.... \\
 & & 1.... \\
 & & 3456(d)
 \end{array}$$

$(o) 394... (n) 4286...$
 $(b) 328... (o) 3940...$
 $722)$
 $346(\downarrow 0,4,$
 $(p) 4236... (n) 42864...$
 $(c) 3432... (p) 42368...$
 $7668)$
 $496(\downarrow 0,0,6,$
 $(q) 428301... (n) 428649...$
 $(d) 34547 (q) 428301...$
 $773771)$
 $348(\downarrow 0,0,0,4,$
 $(r) 4286$
 $(d) 3456$
 7742
 $(r) 428611105$ take
 $(n) 428649036$ from
 $7742)$
 $....$
 37931
 $30968 (4$
 6963
 $6194 (8$
 769
 $697 (9$
 72
 $70 (9$

$$\therefore x = 3 \downarrow 1,4,6,4,4,8,9,9, = 3.45620015$$

As an independent and direct solution of the equation $x^x = a$ has not been attempted by any mathematician, the work of this first general and direct solution is given at length, with exception of that which determined $\downarrow 1$; and although this part of the root may be found by mere inspection, yet it may be desirable to find $\downarrow 1$, in a formal manner.

$$\begin{array}{rcll}
 (m) \ 329 \dots * & (n) \ 428 \dots & 3 = 109866750, & \downarrow 428649036, (n) \\
 (a) \ 285 \dots & (m) \ 329 \dots & 3 & \\
 \hline
 614 & 99 (\downarrow 1, & (m) \ 329600250 & 9531497 (A) \\
 & & & 3 \\
 & & & \hline
 & & & 28594491 (a)
 \end{array}$$

18. *Given*

$x^x = 8722 \cdot 83528 = 10^3 \cdot 8 \downarrow 0,8,6,9,0,3,2,0, = \downarrow 907415561,$
to find the value of x .

$$6 = \downarrow 179184951,$$

$$5 = \downarrow 160951879,$$

Hence x must be situated between 5 and 6.

$$\begin{array}{rcll}
 (m) \ 804|7 \dots * & (n) \ 9074 \dots & 160951879 & 907415561 (n) \\
 (a) \ 476|5 \dots & (m) \ 8047 \dots & 5 & \\
 \hline
 1281|2 & 1027 & (m) \ 804759395 & 9531497 (A) \\
 & & & 5 \\
 & & & \hline
 & & & 47657485 (a)
 \end{array}$$

Since 1281 is contained in 1027, no times the first part of the root must be $5 \downarrow 0$,

$$\begin{array}{rcll}
 (m) \ 8|04 \dots ** & (n) \ 907 \dots & 160951879 & 907415561, (n) \\
 (b) \ 4|97 \dots & (m) \ 804 \dots & 5 & \\
 \hline
 13|01) & 103 (\downarrow 0,7, & (m) \ 804759395 & 995083 (B) \\
 & & 34827905 & 7(b) \\
 & & \hline
 & & 839587300 \downarrow 7 & 4975415 (b) \\
 & (p) \ 900151226 & &
 \end{array}$$

(*b*), (*m*), and (*n*), being found, the next part of the root ↓ 0,7, becomes known, because 13 is contained in 103 seven times. Seven times (*b*) is then added to (*m*), and the sum multiplied by ↓ 0,7, (written ↓ 7,|) when (*p*) is produced.

(<i>p</i>) 900... *** (c) 535... 1435)	(<i>n</i>) 9074..... (p) 9001..... 73 (↓ 0,0,5, 99955 (C) 5 499775 ↓ 7, 535827 (c)	(p) 900151226 2679135 5 (c) 902830361 ↓ 5, (q) 907353550
---	--	--

(*c*) being found, and (*n*) and (*p*) being known, the next figure of the root is found to be ↓ 0,0,5, (written ↓ |5,|); then 5 (*c*) is added to (*p*), and the sum multiplied by ↓ |5,| which gives (*q*).

(q) 9073 . ***** (d) 5387 . 14460 .	(n) 9074 155.. (q) 9073 535.. 620.. ↓,0,0,0,0, 10000 (D) 5 50000 ↓ 0,7,5, 53876 (d)	(q) 907353550
---	---	----------------------

This step gives ↓ 0,0,0,0, for the next part of the root, which is now determined as far as 5 ↓ 0,7,5,0,

(q) 9073.***** (e) 5388 14461)	(n) 9074155.. (q) 9073535.. 620.. (↓ 4, 1000 (E) 5 5000 ↓ 0,7,5,0, 5388 (e)	(q) 907353550 21552 4 (e) 907375102 ↓ 4, (r) 907411402
---	---	---

It is evident that the divisor 14461 remains constant, and that the remainder of the root may be found by common division.

$$\begin{array}{r}
 \begin{array}{l} (n) \quad 907415561, \\ (r) \quad 907411402 \\ \hline 14461 \quad \cdot \cdot \cdot \dagger \end{array}
 \begin{array}{r}
 4159 \quad (2, \\
 2892 \\
 \hline
 1267 \quad (8, \\
 1156 \\
 \hline
 111 \quad (8, \\
 115 \\
 \hline
 \end{array}
 \end{array}$$

$$\therefore x = 5 \downarrow 0,7,5,0,4,2,8,8, = 5.38776485$$

The operations of each step are particularized to show that the process is conducted with certainty, and not depending on conjecture.

19. *Required the value of x in the equation*

$$(\sin x)^{\log x} = 2.69919746 = 2 \downarrow 3,1,3,9,2,8,7,2, = \downarrow 99300512, \quad (n),$$

supposing the logarithms to be hyperbolic and the radius = 10.

Put $e = 100005025$, $\downarrow \overline{x}, 1 = x$, $\downarrow \overline{s}, 1 = \sin x$, reduced to the extent of (n) or $\downarrow 99300512, 1$; and let $1 \wedge 3$ represent the continued product $1.2.3$, $1 \wedge 5$ the continued product $1.2.3.4.5$, and so on.

Then because $(\log. x) (\log. \sin x) = \log. n$.

$$\therefore \frac{\downarrow \overline{x}, 1}{e} \times \frac{\downarrow \overline{s}, 1}{e} = \frac{n}{e}$$

$$\text{or } \downarrow \overline{x}, 1 \times \downarrow \overline{s}, 1 = n \times e = 9930552, (m).$$

e

In the sequel it will be shown that

$$\downarrow \overline{x}, | = \downarrow \overline{100274740}, | = \downarrow 0, 0, 2, 7, 4, 4, 6, 7,$$

$$\text{and } \downarrow \overline{s}, | = \overline{99033417}, | ; \text{ then}$$

$$99033417 \downarrow 0, 0, 2, 7, 4, 4, 6, 7, = 99305502. (m)$$

Since $\downarrow \overline{x}, | \times \downarrow (x - \frac{x^3}{1 \frown 3} + \frac{x^5}{1 \frown 5} - \frac{x^7}{1 \frown 7} + \dots), |$
 $= 99305502, (m)$ it is easily observed that the square root of
 $(m), = 99652110$, is very little greater than $\downarrow (x - \frac{x^3}{1 \frown 3} + \frac{x^5}{1 \frown 5} - \dots), |$
 but not as great as $\downarrow \overline{x}, |$. If $\downarrow \overline{100000000}, |$ be taken $= \downarrow \overline{x}, |$ the
 corresponding value of $\downarrow (x - \frac{x^3}{1 \frown 3} + \frac{x^5}{1 \frown 5} - \dots), |$ will be
 found $= 98770506$, then

$$\begin{array}{r} 993 \dots\dots \\ 987 \dots\dots \\ \hline 1980 \end{array}$$

$$\begin{array}{r} 993 | 055 | 02 \quad (m) \\ 987 | 705 | 06 \quad (r) \\ \hline \end{array}$$

$$\begin{array}{r} 5 | 349 | 96 \quad (\downarrow 0, 0, 2, 7, \\ 3 \ 960 | \\ \hline 1 \ 389 \\ 1 \ 386 \\ \hline \end{array}$$

Putting $\downarrow \dots$ for the required expression, then because ultimately

$$r \downarrow \dots = \frac{m}{\downarrow \dots}$$

$$\text{or } r + \downarrow \dots = m - \downarrow \dots$$

$$\therefore 2 \downarrow \dots = m - r$$

which establishes the method of obtaining the consecutive numbers of the expression $\downarrow 0, 0, 2, 7, \dots$; the whole of this result might be employed as part of the value of $\downarrow \overline{x}, |$, but for the purpose of illustration, only $\downarrow 0, 0, 2$, will be next employed.

$$100000000 \downarrow 0,0,2, = \downarrow \overline{100200100}, = 2 \downarrow 3,2,2,9,7,3,3,2, \\ = 272358946.$$

$$\begin{array}{r} + 100200100 \\ 10 = - 230270081 \\ \hline - 130069981 \\ \hline 3 \end{array}$$

$$\begin{array}{r} - 390209943 \\ 1.2.3 - 179184951 \\ \hline - 569394894 \\ + 690810243 \end{array}$$

$$121415349 = \cdot 003 \downarrow 1,2,0,2,6,9,3,6, = \cdot 003367237$$

$$\begin{array}{r} - 130069981 \\ \hline 5 \\ - 650349905 \\ 1 \curvearrowright 5 = - 478773232 \end{array}$$

$$\begin{array}{r} - 1129123137 \\ 10^5 = + 1151350405 \end{array}$$

$$22227268 = \cdot 00001 \downarrow 2,3,1,7,9, \dots = \cdot 000012489$$

$$\begin{array}{r} - 130069987 \\ \hline 7 \\ - 910489867 \\ 1 \curvearrowright 7 = - 852558978 \end{array}$$

$$\begin{array}{r} - 1763048845 \\ 10^8 = + 1842160648 \end{array}$$

$$79111803 = \cdot 00000002 \downarrow 1,0,2, \dots = \cdot 000000022$$

$$\begin{array}{r} \cdot 272358946 + \\ \cdot 003367237 - \\ 12489 + \\ 22 - \end{array}$$

$$269004176 = 2 \downarrow 3,1,0,5,2,8,7,0, = \downarrow \overline{98969045},$$

$$98960645 \downarrow 0,0,2, = 9915|8665 \text{ take } 9930|5502 (m) \text{ from}$$

$$\begin{array}{r} 9915 \dots\dots \\ 9930 \dots\dots \\ \hline 19845 \dots \end{array} \quad \begin{array}{r} 14|6837 \\ 13|8915 \\ \hline 7922 \\ 7938 \\ \hline \end{array} \quad (\downarrow 0,0,0,7,4,$$

Again, if $\downarrow 0,0,2,7, = \downarrow 100270261,$ be substituted for $\downarrow \bar{x},$ the result 99296696 is obtained; then

$$\begin{array}{r} 99305 \dots \\ 99296 \dots \\ \hline 198601 \\ \dagger \end{array} \quad \begin{array}{r} 99305|502 (m) \\ 99296|696 \\ \hline 8|806 \\ 7|944 \\ \hline 862 \\ 794 \\ \hline \end{array} \quad (\downarrow 0,0,0,0,4,4,$$

If greater accuracy be required, $\downarrow 0,0,2,7,4,4,$ when substituted for $\downarrow \bar{x},$ gives $\downarrow 002,7,4,4,6,7, = 100274740,$ the value of $\downarrow \bar{x},$ to nine places of figures.

But $\downarrow 100274740, = 2 \downarrow 3,2,3,7,2,0,1,7, = 2.725623,$ which is the length of an arc of $15^\circ 37'$ to radius 10.

DUAL ARITHMETIC.

PART I.

DEFINITIONS AND ELEMENTARY PROPOSITIONS.

BECAUSE this system of Arithmetic requires numbers to be viewed under two aspects, and to distinguish it from other systems of operating upon numbers, I have called it DUAL ARITHMETIC. By this new art, a number representing any given magnitude, or the function of any given magnitude, may be made to assume a form composed of factors of whole numbers having a known relation to one another; and these derived whole numbers may be readily made to assume a variety of forms, each form always reducible to the given number or magnitude; and hence the derived numbers, by a peculiar arrangement, may be developed to suit different operations; and the factors produced after such operations are performed, are easily converted into natural numbers expressing the required results.

The more general development of this system will be given in the Author's works on Algebra and the Calculus; in the propositions that follow, the object is rather to establish new Arithmetical processes than to perform operations with conciseness.

PROPOSITION I.

TO MULTIPLY ANY GIVEN NUMBER BY ANY GIVEN POWER OF
1.1, 1.01, 1.001, 1.0001, 1.00001, ETC.

When the powers are whole numbers, the multiplication is performed by the aid of the co-efficients of any binomial $(x + y)$ raised to the proposed power.

$(x + y)^1 = x + y$, the co-efficients are 1, 1.

$(x + y)^2 = x^2 + 2xy + y^2$, the co-efficients are 1, 2, 1.

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$, the co-efficients are 1, 3, 3, 1.

The co-efficients of $(x + y)^4$ are 1, 4, 6, 4, 1.

„ „ $(x + y)^5$ „ 1, 5, 10, 10, 5, 1.

„ „ $(x + y)^6$ „ 1, 6, 15, 20, 15, 6, 1.

„ „ $(x + y)^7$ „ 1, 7, 21, 35, 35, 21, 7, 1.

„ „ $(x + y)^8$ „ 1, 8, 28, 56, 70, 56, 28, 8, 1.

„ „ $(x + y)^9$ „ 1, 9, 36, 84, 126, 126, 84, 36, 9, 1.

The co-efficients of $(x + y)$, in any power, say the 20th, may be at once set down, without knowing the co-efficients of any other power, thus: the co-efficient of the first term is 1, and that of the second term 20, or

$x^{20} + 20x^{19}y$, are the two first terms of the development,

$\frac{20 \times 19}{2} = 190$, the co-efficient of the third term;

hence the three first terms of the development are

$$x^{20} + 20x^{19}y + 190x^{18}y^2.$$

Again, to find the co-efficient of the fourth term,

$$\frac{190 \times 18}{3} = 1140, \text{ co-eff. of 4th term.}$$

$$\frac{1140 \times 17}{4} = 4845, \text{ co-eff. of 5th term.}$$

$$\frac{4845 \times 16}{5} = 15504, \text{ co-eff. of 6th term.}$$

In practice the co-efficients of such high powers are seldom required, but it will be found convenient to be able to set down the co-efficients without being obliged to refer to tables.

$$1, 10, \frac{10 \times 9}{2} = 45, \frac{45 \times 8}{3} = 120, \frac{120 \times 7}{4} = 210,$$

are the first five co-efficients of $(x + y)^{10}$, and set down with little mental exertion.

This method of finding the co-efficients of

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{1 \cdot 2} x^{n-2}y^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^{n-3}y^3 + \&c.$$

will be found more convenient, than by direct substitution. For example, when $n = 10$, the co-efficient of the 6th term, according to the series just given, is

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 252.$$

Examples.

1. Let it be required to multiply 54247 by $(101)^6$, true to ten places of figures.

The number must be divided into

Single digits, when the multiplier is	11,
Periods of two figures	101,
„ three	1001,
„ four	10001,
and so on.	

e	d	c	b	a		
54	24	70	00	00	~ 1	
3	25	48	20	00	~ 6	beginning at a
	8	13	70	50	~ 15	b
		10	84	94	~ 20	c
			8	14	~ 15	d
				3	~ 6	e

$$54247 \times (101)^6 = 57\,58\,42\,83\,61, \text{ true to ten places of figures.}$$

Since the multipliers for the 6th power are

$$1, 6, 15, 20, 15, 6, 1;$$

begin at a , a period in advance, and multiply by 6; then commence at b , two periods in advance, and multiply by 15; at c , three periods in advance, and multiply by 20; at d , four periods in advance (counting from the right to the left), and multiply by 15; the period e should be multiplied by 6, but, as it is blank, only set down 3, which is obtained by multiplying d , or rather the first figure of d , by 6.

As it is very easy to operate with the co-efficients

$$1, 5, 10, 10, 5, 1,$$

the multipliers for the 5th power, it may be more convenient to multiply the given number by $(101)^5$, and then by $(101)^1$.

To multiply any number, as 54247, by 5, affix a cypher, or suppose one affixed, half this number will be 5 times the given one,

$$\begin{aligned} \text{half of } 542470 &= 271235 = \\ &5 \text{ times } 54247. \end{aligned}$$

e	d	c	b	a			
54	24	70	00	00			~ 1
27	123	50	00				~ 5 beginning at a
	542	47	00				~ 10 „ b
	542	47					~ 10 „ c
		27	1				~ 5 „ d
			1				~ 1 „ e

$$54247 \times (101)^5 = \begin{array}{r} 57 \mid 01 \mid 41 \mid 42 \mid 19 \\ \hline 57 \mid 01 \mid 41 \mid 42 \end{array}$$

$$\underline{\underline{57 \ 58 \ 42 \ 83 \ 61}} = 54247 \times (101)^6.$$

To render the operation clear, the decimal points are omitted, but the result is easily pointed, for

$$\begin{aligned} \cdot 54247 \times (1\cdot 01)^6 &= \cdot 5758428361, \\ 54\cdot 247 \times (1\cdot 06)^6 &= 57\cdot 58428361, \\ \&c. &= \&c. \end{aligned}$$

The correctness of such results may be proved by common multiplication.

The multipliers for the different powers may be operated with in many ways: the present example will illustrate this remark:

6	15	20	15	6	1		
<i>e</i>	<i>d</i>	<i>c</i>	<i>b</i>	<i>a</i>			
	54	24	70	00	00	~	1
	3	25	48	20	00	~	6 . . . <i>a</i>
		8	13	70	50	~	15 . . . <i>b</i>
			10	84	94	~	20 . . . <i>c</i>
				8	14	~	15 . . . <i>d</i>
					3	~	6 . . . <i>e</i>

325482000 is found by multiplying the first line by 6, beginning with the period *a*; the next line may be obtained by multiplying the first line by 15, beginning at the period *b*, but it is known that 6 times the first line, beginning at the period *b*, is 3254820; 60 times it must therefore be 32548200, the quarter of which is 8137050 = 15 times the first line, beginning at the period *b*; so that the 15 line is found by little more than dividing the 6 line by 4. 20 times the first line, beginning at the period *c*, is found by doubling the first line, and adding a cypher. It seldom happens, in practice, that more than seven or eight places of figures are required, but these elementary examples are extended to a greater number of places, to make the simplicity of the method more apparent. Different plans of obtaining the same result, without altering the laws that govern the system, give proof and dexterity to the operator.

Without changing the example, let it be required to multiply by another method, 54247 by $(101)^6$, true to 12 places of figures.

54	24	70	00	00	00	*	put = A	
3	25	48	20	00	00		$6 \times A = B$, beginning under *	
	8	13	70	50	00		$\frac{5 \times B}{2} = C$,	" *
		10	84	94	00		$\frac{4 \times C}{3} = D$,	" *
			8	13	71		$\frac{3 \times D}{4} = E$,	" *
				3	25		$\frac{2 \times E}{5} = F$,	" *
					1		$\frac{1 \times F}{6} = G$,	" *

57 58 42 83 60 97, true to 12 places of figures.

If only six places of figures are required, the work is short and the calculations simple :

		*	
54	24	70	Put = A
3	25	48	$6 \times A \div 1 = B$
	8	14	$5 \times B \div 2 = C$
		11	$4 \times C \div 3 = D$
<u>57 58 43</u>			

$$\begin{aligned}
 5424 \cdot 7 \times 6 &= 32548 \cdot 2. \\
 325 \cdot 48 \times 5 \div 2 &= 813 \cdot 70, \text{ put down } 814. \\
 8 \cdot 14 \times 4 \div 3 &= 10 \cdot 85, \text{ put down } 11.
 \end{aligned}$$

The next line is rejected, because the result obtained would not increase the required product a unit in sixth place of figures, reckoning from the left to the right.

c	b	a	
54	24	70	~ 1
3	25	48	~ 6 times, beginning at the period a
	8	14	~ 15 " " "
		11	~ 20 " " "

$$\begin{aligned}
 5424 \cdot 7 \times 6 &= 32548 \cdot 2 \\
 54 \cdot 24 \times 15 &= 813 \cdot 6, \text{ put down } 814. \\
 54 \times 20 &= 10 \cdot 80, \text{ put down } 11.
 \end{aligned}$$

2. Multiply 34567812 by $(10001)^8$, so that the result may be true to 12 places of figures.

The multipliers for the 8th power are 1, 8, 28, 56, 70, 56, 28, 8, 1.

$$\begin{array}{r|l|l|l|l|l}
 d & c & b & a & & \\
 \hline
 & 3456 & 7812 & 0000 & \dots\dots & \sim 1 \\
 & 2 & 7654 & 2496 & \dots\dots & \sim 8 \text{ times, } a \\
 & & 9 & 6790 & \dots\dots & \sim 28 \text{ times, } b \\
 & & & 19 & \dots\dots & \sim 56 \text{ times, } c \\
 \hline
 & 3459 & 5475 & 9305 & &
 \end{array}$$

A blank period of dots may be affixed to facilitate the operation ; with dots 56 times the first line, beginning at c , is thus found, 56 times 6 gives a dot, 56 times 5 gives a dot, 56 times 4 gives a dot and carry 22, 56 times 3 gives 168, to which add 22 = 190, put down a dot, and place 19 under the next period.

The work is very easy when 28 times is found, as 9·67, &c. is found for 28 times, the double of which is 19·34, put down 19.

28 times the first line, beginning at b , is readily found, when 8 times is known. Double the first line, with 0 affixed, gives 691356240, putting dots for 4 of the figures.

$$\begin{array}{rcl}
 69136 & \dots\dots & 20 \text{ times, beginning at } b \\
 27654 & \dots\dots & 8 \quad \text{,,} \quad \text{,,} \quad b \\
 \hline
 96790 & \dots\dots & 28 \quad \text{,,} \quad \text{,,} \quad b
 \end{array}$$

Hence, when 8 times is found, 28 times and 56 times may be obtained by little more than inspection. However, results are seldom required to more than 7 or 8 places of figures ; if the product of 34567812 by $(10001)^8$ is only required to 7 places of figures, the work is much contracted, and may be arranged as follows :—

$$\begin{array}{r|l|l|l|l}
 & * & & & \\
 \hline
 3456 & 7812 & \dots\dots & \sim \text{Put} = A. \\
 2 & 7654 & \dots\dots & \sim 8 \times A \div 1 = B, \text{ beginning under } * \\
 & 10 & \dots\dots & \sim 7 \times B \div 2 = C, \quad \text{,,} \quad * \\
 \hline
 3459 & 548, & \text{true to 7 places of figures.}
 \end{array}$$

$8 \times A \div 1$, beginning under $*$, is worked thus : 8 times 2 gives a dot, 8 times 1 gives a dot, 8 times 8 gives a dot, but

carry 6; 8 times 7 are 56, and 6 are 62, put down a 4th dot and carry 6; then 8 times 6 are 48, and 6 are 54, put down 4 and carry 5, and continue the multiplication in the common way. $7 \times B \div 2$, beginning under *, is readily found, thus: 7 times 4 gives a dot, 7 times 5 gives a dot, 7 times 6 gives a dot, but carry 4; 7 times 7 and 4 gives a dot, but carry 5; 7 times 2 and 5 = 19, the half of which is 9.5 and may be counted 10. So that but little mental labour is employed, although it takes many words to explain the operation.

To find the first 12 figures of the product of 34567812 by $(10001)^8$, the remaining multipliers, 70, 56, 28, 8, 1, are not required. Perhaps the required result might be obtained with greater ease by first multiplying 34567812 by $(10001)^5$, and the product thus found by $(10001)^3$; by this plan the work may be arranged as follows:—

$$\begin{array}{r|l|l|l|l}
 c & b & a & & \\
 \hline
 3456 & 7812 & 0000 & \sim & 1 \\
 & 17283 & 9060 & \sim & 5, a \\
 & & 34568 & \sim & 10, b \\
 & & & 3 & \sim 10, c
 \end{array}$$

$$\begin{array}{r|l|l|l|l}
 3458 & 5099 & 3631 & \sim & 1, \\
 & 10375 & 5298 & \sim & 3, a \\
 & & 10376 & \sim & 3, b \\
 & & & 0 & \sim 1, c
 \end{array}$$

3459 5475 9305, result as before.

With a blank period of dots the work will stand thus:

$$\begin{array}{r|l|l|l|l}
 c & b & a & & \\
 \hline
 3456 & 7812 & 0000 & \dots \sim & 1 \\
 & 17283 & 9060 & \dots \sim & 5, a \\
 & & 34568 & \dots \sim & 10, b \\
 & & & 3 & \dots \sim 10, c
 \end{array}$$

$$\begin{array}{r|l|l|l|l}
 3458 & 5099 & 3631 & \dots \sim & 1 \\
 & 10375 & 5298 & \dots \sim & 3, a \\
 & & 10376 & \dots \sim & 3, b \\
 & & & 0 & \dots \sim 1, c
 \end{array}$$

3459 5475 9305

3. *Required the continued product of 1.2345678, $(1.01)^4$, $(1.001)^5$, and $(1.0001)^6$, true to 7 places of figures.*

$$\begin{array}{r|l} 12 & 345678 \sim 1 \\ & 493827 \sim 4 \\ & 7407 \sim 6 \\ & 49 \sim 4 \end{array}$$

$$\begin{array}{r|l} 128 & 4696 \dots \sim 1 \\ & 6423 \dots \sim 5 \\ & 13 \dots \sim 10 \end{array}$$

$$\begin{array}{r|l} 1291 & 132 \dots \sim 1 \\ & 775 \dots \sim 6 \\ & 0 \dots \sim 15 \end{array}$$

$$\underline{1.291907} = \text{the first seven figures of the}$$

continued product of $1.2345678 \times (1.01)^4 \times (1.001)^5 \times (1.0001)^6$; which may be written,

$$1.2345678 \downarrow 0,4,5,6, = 1.291907.$$

The arrow $\downarrow$ divides the co-efficient 1.2345678 and the powers of 1.1, 1.01, 1.001, 1.0001, &c. 0, immediately follows the arrow, because no power of 1.1 is employed; 6, is in the fourth place after $\downarrow$, and shows that this power operated upon periods of four figures each; 5 being in the third place after $\downarrow$, shows, by its position, that its influence is over periods of three figures each; and 4 occupies the second place after $\downarrow$, and reminds the operator that its influence is over periods of two figures each.

4. *Required the first 9 figures of the continued product of 32.808992, $(1.01)^7$, and $(1.001)^9$; or, according to the notation just adopted, the value of*

$$32.808992 \downarrow 0,7,9,$$

is required to nine places of figures.

Dots take the place of figures not required in the last period.

<i>e</i>	<i>d</i>	<i>c</i>	<i>b</i>	<i>a</i>		
	32	80	89	92	0 .	1
	2	29	66	29	4 .	7 times, beginning at <i>a</i>
		6	88	98	9 .	21 " " <i>b</i>
			11	48	3 .	35 " " <i>c</i>
				11	5 .	35 " " <i>d</i>
					1 .	21 " " <i>e</i>
	35 17 56 80 2					

The way in which this result may be found is worth noting; but little mental labour is necessary, the co-efficients or multipliers being

1, 7, 21, 35, 35, 21, 7, 1,

when 7 times the first line (commencing with the period marked *a*) is found, 21 times the same line (commencing with the period marked *b*) may be determined by multiplying the second line by 3, beginning under *a*. Again, 35 times the first line, commencing with the period *c*, gives the same result as multiplying the second line by 5, commencing under *b*; but 5 times a number may be found by moving the decimal point one figure to the right, and then take the half.

22966.3 $\sim$ 7 times, beginning at *b*.
 half = 114831, $\sim$ 35 times.
 = 11483. putting a dot for the figure not required. Since the example requires that 351756802, is to be again multiplied by $(1.001)^9$, the remainder of the work may stand as follows:—

<i>c</i>	<i>b</i>	<i>a</i>		
	351	756	802	$\sim$ 1,
	3	165	811	$\sim$ 9 beginning at <i>a</i>
		12	663	$\sim$ 36 " <i>b</i>
			30	$\sim$ 84 " <i>c</i>
	354 935 306			

The 9 times line, beginning at *a*, may be found by subtraction, thus,

351756·8 has to be multiplied by 9;

3517568 ten times

351757 once

3165811 9 times.

The line 36 times begins at *b*, observing to carry from the preceding figure as if it were a decimal, and making the usual allowance when the number is followed by 5, 6, 7, 8, or 9.

$$351·76 \times 36 = 12663 = 3165·8 \times 4.$$

To multiply by 84, beginning at *c*: the period *c*, or ·351, may be called ·352, as 7 follows the last figure,

$$84 \times \cdot 352 = 29·568 = 30,$$

after the usual allowance is made.

Or the work may stand thus:—

351	*	756	802	put = A
3	165	811		9 × A ÷ 1 = B beginning under *
	12	663		8 × B ÷ 2 = C " "
		30		7 × C ÷ 3 = D " "
354935306				

$$\therefore 32·808992 \downarrow 0,7,9, = 35·4935306.$$

In operating with the co-efficients or multipliers many contractions will suggest themselves to the operator.

5. *Required the first 7 figures of the continued product of 106, (1·1)², (1·01)⁶, (1·001)⁷, (1·0001)¹, (1·00001)⁶, and (1·0000001)²; or, which is the same thing, find the first 7 figures of 106 ↓ 2,6,7,1,6,2,*

$$\begin{array}{r} |1|0|6|0|0|0|0 \\ |2|1|2|0|0|0|0 \\ |1|0|6|0|0|0|0 \\ \hline \end{array}$$

$$\begin{array}{r} |1|2|8|2|6|0|0| \\ |7|6|9|5|6| \\ |1|9|2|4| \\ |2|6| \\ \hline \end{array}$$

$$\begin{array}{r} |1|3|6|1|5|0|6|.. \\ |9|5|3|1|.. \\ |2|9|.. \\ \hline \end{array}$$

$$\begin{array}{r} 1|3|7|1|0|6|6| \\ |1|3|7| \\ \hline \end{array}$$

$$\begin{array}{r} 1|3|7|1|2|0|3|... \\ |8|2|... \\ \hline \end{array}$$

$$\begin{array}{r} 1|3|7|1|2|8|5|..... \\ \hline \end{array}$$

$$\begin{array}{r} 1|3|7|1|2|8|5|..... \\ |3|..... \\ \hline \end{array}$$

$$\begin{array}{r} 1|3|7|1|2|8|8| \\ \hline \end{array}$$

$\therefore 106 \downarrow 2,6,7,1,6,2 = 1371288$, true to 7 places of figures as required. To balance the last period dots may be put to represent numbers; this plan involves no additional labour, but secures accuracy. Operating for the last 2, which is in the sixth place, 2 complete periods of 6 figures each are represented by placing 5 dots after 1371285; it then becomes,

$$\begin{array}{r} \text{Q} \quad \text{P} \\ 137128|5|..... \\ |10|..... \end{array}$$

Suppose the period of 6 figures Q, had to be multiplied by 7, and placed under the period P:—say, 7 times 8 is something to which carry something, for which put down a dot; and then say, 7 times 2 is something to which carry something, and, regardless of the result, put down a second dot; then 7 times 1, to which some number may have to be carried, the result is again disregarded, and only a third dot is set down; 7 times 7,

&c. only gives a fourth dot; 7 times 3, with all carried, only produces a fifth dot; but, because the number 5 requires a number under it:—7 is multiplied by 1, to which 3 is carried, making the result 10, which is set down. 7 times 3 is only 21, because something has to be carried that makes the result more than 25. If 2 were carried, 9 should be placed under the 5; 9 is too small, and the number set down, 10, is too great, yet the true result is nearer to 10 than it is to 9; but in either case the difference is not a unit in the seventh place. So that, to multiply the period marked Q, and place the result under P, is reduced to the simple operation of making five dots, and saying 7 times $1 + 3 = 10$. Those who wish to understand what follows, and operate with ease, should be well acquainted with this process of operating with dots.

While explaining the Elements of Dual Arithmetic, the work is spread out, to render the explanations clear; in practice, however, the numbers employed may be set down in a very compact form. The last example may stand thus:—

$$\begin{array}{r|l} 1 & 060000 \\ & 212000 \\ & 106000 \end{array}$$

$$\begin{array}{r|l} 12 & 82600. \\ & 76956. \\ & 1924. \\ & 26. \end{array}$$

$$\begin{array}{r|l} 136 & 1506.. \\ & 9531.. \\ & 29.. \end{array}$$

$$\begin{array}{r|l} 1371 & 066. \\ & 137.... \\ & 82..... \\ & 3..... \end{array}$$

$$\underline{1371288} = 106 \downarrow 2,6,7,1,6,2,$$

When a period takes up half the required number of figures, or more than half, then one of the co-efficients or multipliers is

only required, and that multiplier is the index of the power operated with.

6. Required the first 8 figures of $3 \downarrow 3,4,5,6,7,8,9,10$, when developed.

$$\begin{array}{r} 3 \overline{) 00000000} \\ 9 \overline{) 00000000} \\ 9 \overline{) 00000000} \\ 3 \overline{) 000000} \end{array}$$

$$\begin{array}{r} 3993 \overline{) 0000} \\ 1597 \overline{) 200} \\ 2395 \overline{) 8} \\ 160 \end{array} = 3 \downarrow 3,$$

$$\begin{array}{r} 415 \overline{) 51318} \\ 2077 \overline{) 57} \\ 416 \end{array} = 3 \downarrow 3,4,$$

$$\begin{array}{r} 4175 \overline{) 9491} \\ 25055 \dots \\ 2925 \dots \\ 334 \dots \\ 37 \dots \\ 4 \dots \end{array} = 3 \downarrow 3,4,5,$$

6,
7,
8,
9,
10,

$$\underline{41787846} = 3 \downarrow 3,4,5,6,7,8,9,10,$$

Every figure employed in this operation is set down; 6 numbers added together produce 41787846; the first partition line, to the left, divides 4175/9491 into periods of 4 figures each, then the number 25055 is found in the usual way. The second partition line divides

$$\begin{array}{r} 41759491 \\ 25055 \dots \end{array}$$

into 2 periods of 5 figures each, counting the dots as 2 figures; the third partition line divides

$$\begin{array}{r} 41759491 \\ 25055 \dots \\ 2925 \dots \end{array}$$

8,

into 2 periods of 6 figures each, counting the 4 dots as figures. In this case, and in cases of similar nature, the sum is not required. In operating at the step above exhibited, the left hand period has to be multiplied by 8, and the result placed under the right hand period; 8 times the sum of 9,0,4, and all that has to be carried, produce only a dot; 8 times the sum of 2,5,9, and all to be carried from the last operation, produce only another dot; 8 times the sum of 2,5, with what has to be carried, produce a third dot; 8 times 7 = 56, so that it is safe to place down a fourth dot and carry 6; then 8 times 1 + 6 = 14, put down 4 and carry 1; 8 times 4 + 1 = 33; the number 334 is found with little mental labour when the partition line and the required dots are properly placed:

$$\begin{array}{r}
 41759491 \\
 25055 \dots \\
 29 \overline{) 25} \dots \dots \\
 3 \overline{) 34} \dots \dots \quad 8,
 \end{array}$$

PROPOSITION II.

TO FIND THE POWERS OF (1·1), (1·01), (1·001), (1·0001), ETC.
 SO THAT WHEN ONE GIVEN NUMBER IS MULTIPLIED BY
 THEM, THE CONTINUED PRODUCT WILL BE ANOTHER GIVEN
 NUMBER.

Examples.

1. What powers of (1·1), (1·01), (1·001), &c. must 23 be continually multiplied by, so that the continued product may be 2345678?

$$\underline{23\ 4\ 5\ 6\ 7\ 8.}$$

$$\begin{array}{r} 23 \overline{) 00\ 00\ 0.} \\ \underline{23\ 00\ 0.} \end{array}$$

$$\begin{array}{r} 23\ 2 \overline{) 3\ 00\ 0.} = 23 \downarrow 0,1, \\ \underline{2\ 09\ 0\ 7.} \\ 8\ 4. \end{array}$$

$$\begin{array}{r} 23\ 4\ 3 \overline{) 99\ 1.} = 23 \downarrow 0,1,9, \\ \underline{1\ 64\ 1.} \end{array}$$

$$\begin{array}{r} 23\ 4\ 5\ 6 \overline{) 3\ 2\ .} = 23 \downarrow 0,1,9,7, \\ \underline{4\ 7\ .} \end{array}$$

$$\underline{23\ 4\ 5\ 6\ 7\ 9} = 23 \downarrow 0,1,9,7,2,$$

$$\therefore 2345678 = 23 \downarrow 0,1,9,7,2,$$

To find any of the operating figures as $\downarrow 0,0,9$,

$$\text{From } 23 \overline{) 456 \dots}$$

$$\text{Take } 23 \overline{) 230 \dots}$$

$$\begin{array}{r} 23) \overline{) 226 \dots} (\downarrow 0,0,9, \\ \underline{207} \end{array}$$

$23 \downarrow 0,1$, shows that no periods of single figures are involved.

2. *Required the multipliers that will bring 880091 to 886327.*

$$\text{Put } n = \begin{array}{r} 880 \overline{) 091} \\ \underline{6\ 160} \\ 18 \end{array}$$

$$886\ 269 \dots = n \downarrow 0,0,7,$$

It is evident, that if the left hand period 8862, be placed under the right one 69 .. and added, the sum would be greater than 886327, hence no operation has to be performed on periods of 4 figures.

$$\begin{array}{r} 88626 \overline{) 9 \dots} = n \downarrow 0,0,7 \\ \underline{5\ 3 \dots} \quad \quad \quad 0,6 \\ \underline{5 \dots} \quad \quad \quad 6, \end{array}$$

$$\underline{88632\ 7} = n \downarrow 0,0,7,0\ 6,6,$$

$$\therefore 880091 \downarrow 0,0,7,0,6,6, = 886327.$$

3. Required the multipliers that will bring 663157 to 663312.

$$\begin{array}{r} \text{Put } m = 6631 \overline{) 57} \dots \\ \quad 1 \overline{) 3} 3 \dots \dots m \downarrow 0,0,0,2, \\ \quad \quad 1 \overline{) 9} \dots \dots \quad 3, \\ \quad \quad \quad 3 \dots \dots \quad 5, \\ \hline 6633 \overline{) 12} = m \downarrow 0,0,0,2,3,5, \end{array}$$

$$\therefore \frac{880091 \times 663312}{886327 \times 663157} = \frac{nm \downarrow 0,0,0,2,3,5,}{nm \downarrow 0,0,7,0,6,6,} = \frac{\downarrow 0,0,0,2,}{\downarrow 0,0,7,0,3,1,}$$

It is easily shown that

$$\frac{\downarrow 0,0,0,2,}{\downarrow 0,0,7,0,3,1,} = .993198.$$

$$\begin{array}{r} 1000 \overline{) 000} . \\ \quad 200 . \\ \hline 1 \downarrow 0,0,0,2, = 100 \overline{) 020} 0 \dots + \\ \quad \quad 700 \overline{) 1} \dots - \\ \quad \quad \quad 28 \dots + \end{array}$$

$$\frac{\downarrow 0,0,0,2,}{\downarrow 0,0,7} = 99322 \overline{) 7} \dots +$$

$$\frac{\downarrow 0,0,2,}{\downarrow 0,0,7,0,3,1,} = 993198$$

Division by the powers of 1.1, 1.01, 1.001, &c. will be explained presently; the process differs but little from that of multiplication. The numbers employed in this example are the cosines of the 4 angles, $27^\circ 35' 5''$; $48^\circ 27' 32''$; $28^\circ 20' 48''$; $48^\circ 26' 49''$, to six places of decimals. In the next example, it is proposed to operate on the cosines of the same angles, continued to seven places of decimals.

4. Reduce the compound fraction $\frac{8800909 \times 6633134}{8863271 \times 6631572}$; to powers of $\downarrow 1,; \downarrow 0,1,; \downarrow 0,0,1,; \&c.$

$$\begin{array}{r} 6633 \overline{) 134} \\ \quad 6631 \overline{) 572} \dots \\ \downarrow 0,0,0,2, \quad 1 \overline{) 3} 26 \dots \quad (A). \\ \quad 3, \quad 1 \overline{) 9} \dots \dots \\ \quad \quad 5, \quad 3 \overline{) 3} \dots \dots \\ \quad \quad \quad 6 \quad 4 \dots \dots \end{array}$$

$$\begin{array}{r}
 886\ 32\ 7\ 1 \\
 \hline
 880\ 09\ 0\ 9 \dots \\
 6\ 16\ 0\ 6 \dots \\
 1\ 8\ 5 \dots \\
 \hline
 \downarrow 0,0,7, \quad 886\ 27\ 0\ 0 \dots \\
 0,6, \quad 5\ 3\ 2 \dots \dots \dots (B). \\
 4, \quad 3\ 5 \dots \dots \dots \\
 4, \quad 4 \dots \dots \dots
 \end{array}$$

or

$$\begin{array}{l}
 \downarrow 0,0,0,2,3,5,6, \\
 \downarrow 0,0,7,0,6,4,4, \\
 \downarrow 0,0,0,2,0,1,2, \\
 \downarrow 0,0,7,0,3,
 \end{array} = \text{the given fraction.}$$

The details of the process have been so fully discussed, it is presumed that the work will be understood without entering into further detail.

In (A) and (B), the last additions are omitted, as unnecessary; it is easily observed, that if the additions were performed, the required results 6633134 and 88663271, set down merely to show the numbers to be made up, would be obtained. It is easily shown that the difference of the results (A) and (B), is equal to the given fraction. For, put $n = 6631572$, and $m = 8800909$,

$$\therefore \frac{8800909 \times 6633134}{8863271 \times 6631572} = \frac{mn \downarrow 0,0,0,2,3,5,6,}{nm \downarrow 0,0,7,0,6,4,4,} = \frac{\downarrow 0,0,0,2,0,1,2,}{\downarrow 0,0,7,0,3,},$$

the numerical value of which is easily found, thus,

$$\begin{array}{r}
 100\ 0\ 00\ 0\ 0 \\
 \hline
 20\ 0\ 0 \\
 \hline
 1 \downarrow 0,0,0,2, \quad 100\ 0\ 20\ 0\ 0 \dots \\
 0,1, \quad 1\ 0 \dots \dots \dots \\
 2, \quad 2 \dots \dots \dots \\
 \hline
 100\ 0\ 20\ 1\ 2 \dots + \\
 7\ 00\ 1\ 4 \dots - \\
 2\ 8\ 0 \dots + \\
 1 \dots - \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 \text{Divided by } \downarrow 0,0,7, \quad 99\ 3\ 22\ 7\ 7 \dots + \\
 0,3, \quad 2\ 9\ 8 \dots - \\
 \hline
 \therefore \frac{\downarrow 0,0,0,2,0,1,}{\downarrow 0,0,7,0,3,} = \frac{99\ 3\ 19\ 7\ 9}{99\ 3\ 19\ 7\ 9}
 \end{array}$$

It may be proper here to state, that the co-efficients employed, when dividing by the 7th power, are 1, - 7, + 28, - 84, + &c.

5. *Required the value of the compound fraction*

$$\frac{8126236 \times 9867261}{8124229 \times 9891787}$$

in terms of the powers of $\downarrow 1, ; \downarrow 0,1, ; \downarrow 0,0,1, \&c.$

$8126236 = \text{cosine of } 35^\circ 38' 49''$; $9867261 = \cos 9^\circ 20' 45''$;
 $8124229 = \cos 35^\circ 40'$; $9891787 = \cos 8^\circ 26' 13''$.

$$\begin{array}{r} 8126236 \\ \hline \downarrow 0,0,0,2, \quad 8124 \overline{) 229} . \\ \quad 4, \quad 1 \overline{) 625} . . . \\ \quad 7, \quad 3 \overline{) 25} \\ \quad \quad 5 \overline{) 7} \end{array}$$

$$9891787 . .$$

$$\begin{array}{r} 986 \overline{) 7261} . . . \\ \quad 1 \overline{) 9735} . . \\ \quad \quad 1 \overline{) 0} . . \end{array}$$

$$\begin{array}{r} \downarrow 0,0,2, \quad 9887 \overline{) 006} . \\ \quad \quad 3 \overline{) 955} . \\ \quad \quad \quad 1 . \end{array}$$

$$\begin{array}{r} 4, \quad 98909 \overline{) 02} . . . \\ \quad 8, \quad 7 \overline{) 91} \\ \quad 3, \quad 3 \overline{) 0} \\ \quad 4, \quad 4 \end{array}$$

$$\therefore \frac{\downarrow 0,0,0,2,4,7,}{\downarrow 0,0,2,4,8,3,4,} = \frac{\downarrow 0,0,0,0,0,4,}{\downarrow 0,0,2,2,4,0,4,} .$$

6. *The natural cosine of* $6^\circ 26' 23''.5 = 9936901$; *the cosine of* $37^\circ 39' 49''.5 = 7916103$; *required the value of* 9936901×7916103 , *true to seven places of figures.*

$$9936901 = 99 \downarrow 0,0,3,7,2,2,$$

$$\therefore 7916103 \times 99 \downarrow 0,0,3,7,2,2, = \text{the required product.}$$

To multiply a number by 99 is a simple matter.

$$\begin{array}{rcl} 791610300 & \sim & 100 \text{ times,} \\ 7916103 & \sim & \text{once,} \end{array}$$

$$\text{Difference } \underline{7836942} \dots \sim 99 \text{ times.}$$

$$\therefore 7916103 \times 9936901 = 7836942 \downarrow 0,0,3,7,2,2, = 7866153.$$

WORK.

$$\begin{array}{r} 783 \overline{) 6942} \dots \\ 1 \overline{) 3511} \dots \\ 24 \dots \end{array}$$

$$\begin{array}{r} 7860 \overline{) 477} \dots \\ 5502 \dots \\ 2 \dots \end{array}$$

$$\begin{array}{r} 78659 \overline{) 81} \dots \\ 1 \overline{) 57} \dots \\ 16 \dots \end{array}$$

$$\underline{7866153}$$

7916103×9936901 gives 78661531816803 by common multiplication, and hence the first seven figures of the product are found true to the last figure.

PART II.

DIVISION OF DUAL ARITHMETIC.

THE method of division is more readily established by particular examples than by generalization ; this being granted, let it be required to divide by 1 (1'01)³. The decimal point may be omitted during the operation, and (1'01)³ written (101)³.

The co-efficients of $(x + y)^{-3}$, developed by the binomial theorem, are readily found ; the first term is x^{-3} , the second term is $-3 x^{-4} y$; the co-efficient -3 , the same as the power, and, like it, negative ; x^{-4} has a power (1) less than the power of x^{-3} . $-3 \times -4 \div 2 = +6$, hence the third term will be $+6 x^{-5} y^2$, the index of x is decreased by unity, while the index of y is increased by unity : -3×-4 is divided by 2, because $-3 x^{-4} y$, is the second term, from which is deduced $+6 x^{-5} y^2$, the third term. $+6 \times -5 \div 3 = -10$, the co-efficient of the fourth term, $-10 x^{-6} y^3$; so that $(x + y)^{-3}$ is developed in the same way as $(x + y)^3$.

$$\therefore (x + y)^{-3} = x^{-3} - 3 x^{-4} y + 6 x^{-5} y^2 - 10 x^{-6} y^3 + 15 x^{-7} y^4 - \dots$$

which, when compared with $\left(1 + \frac{1}{100}\right)^{-3} = \frac{1}{(1'01)^3}$, gives

$$\left(1 + \frac{1}{100}\right)^{-3} = 1 - 3 \times \frac{1}{100} + 6 \times \frac{1}{(100)^2} - 10 \times \frac{1}{(100)^3} + \dots$$

observing that every power of 1 is 1. The multipliers or co-efficients for the division by $(1'01)^{-3}$ are

$$1 ; -3 ; +6 ; -10 ; +15 ; - \&c.$$

The co-efficient of the fifth term being $+15$, and -7 the index of x , therefore the co-efficient of the sixth term

$$= +15 \times -7 \div 5 = -21.$$

Though it seldom happens that many of these multipliers or co-efficients are employed, in these elementary examples the work is extended beyond the limits required in practice, so that the laws which govern the operations may be easily detected.

The work may be arranged under the form established in Part I., as follows:—

$+$	e	d	c	b	a		$+$	$1,$	
$-$	10	00	00	00	00	Mult. by	$-$	$3,$	beginning at a
$+$		30	00	00	00		,,	$+$	$6,$
$-$			60	00	00		,,	$-$	$10,$
$+$			1	00	00		,,	$+$	$15,$
$-$				1	50		,,	$-$	$21,$
					2		,,		
970590148									

Sum of the negative terms = 1000600150 by inspection.

Sum of the positive terms = 30010002 „
 970590148

In many cases, the difference of the positive and negative terms may be found without summing each class, and taking the difference. When this simple example is understood, what follows will appear easy. However, it may be asked, how -21 times the first line, beginning under e , produces -2 , as no number exists under e : the -2 is carried from the next term, when multiplied by -21 .

2. Divide 123 by $(10001)^7$, true for the first ten places of figures.

$123 \div (10001)^7$, may be written $123 \downarrow 0,0,\overline{7}$, the negative sign being placed over the figure 7 ; for $123 \div (10001)^7$ equal

$$\frac{123}{(10001)^7} = 123 \times (10001)^{-7}$$

Set down $123 \downarrow 0,0,0,\overline{7}$, for $123 \div (10001)^7$.

The first multiplier for the seventh power is 1, the second - 7,

$$\text{the third} = \frac{-7 \times -8}{2} = +28;$$

$$\text{the fourth} = \frac{+28 \times -9}{3} = -84;$$

$$\text{the fifth} = \frac{-84 \times -10}{4} = +210;$$

&c.

&c.

	<i>c</i>	<i>b</i>	<i>a</i>	
+	1230	0000	00 ..	+ 1,
-		8610	00 ..	- 7, beginning at <i>a</i>
+		3	44 ..	+ 28, " <i>b</i>
-			..	- 84, " <i>c</i>
1229 1393 44				

This result is instantly found, as $28 = 4$ times 7, and $84 = 3$ times 28; but the multiplier - 84, gives a result that does not affect the first ten places of figures, and is therefore omitted.

$$\therefore 123 \downarrow 0,0,0,7, = 1229139344.$$

As in multiplication, when the factors are large, each step is readily obtained from the one preceding, thus:

		*		
+ 1230	0000	00 ..	call	+ A
-	8610	00 ..		- 7 + A ÷ 1 = - B begin under *
+	3	44 ..		- 8 × - B ÷ 2 = + C begin under *
				.. - 9 × + C ÷ 3 = - D begin under *
1229 1393 44				

	<i>h</i>	<i>g</i>	<i>f</i>	<i>e</i>	<i>d</i>	<i>c</i>	<i>b</i>	<i>a</i>	
+	6	4	3	6	0	0	0	0	Mult. + 1 begin at
-	1	2	8	7	2	0	0	0	- 2 „ <i>a</i>
+		1	9	3	0	8	0	0	+ 3 „ <i>b</i>
-			2	5	7	4	4	0	- 4 „ <i>c</i>
+				3	2	1	8	0	+ 5 „ <i>d</i>
-					3	8	6	2	- 6 „ <i>e</i>
+						4	5	1	+ 7 „ <i>f</i>
-							5	1	- 8 „ <i>g</i>
+								6	+ 9 „ <i>h</i>

66323437, sum of positive terms.

13133353, sum of negative terms.

$$\text{Quotient} = \underline{53190084} = 6436\downarrow\bar{2},$$

The quotient just found may be shown to be correct by multiplying it by $(11)^2$, according to the methods explained in Part I. The multipliers for the square being 1, 2, 1;

$$\begin{array}{r} 53190084 - 1 \\ 10638016 - 2 \\ \hline 531900 - 1 \end{array}$$

Proof, 64360000

5. Divide 3141593 by $(101)^1$, and give the first seven figures of the quotient.

The multipliers for the first power are

$$+ 1; - 1; + 1; - 1; + 1; \&c.$$

$$\begin{array}{r} + 3141593 \cdot + 1 \\ - 31416 \cdot - 1 \\ + 314 \cdot + 1 \\ - 3 \cdot - 1 \end{array}$$

3141907, sum of positive terms.

31419, sum of negative terms.

$$\therefore \underline{3110488} = 3141593\downarrow\bar{0,1},$$

To divide by 11, 101, 1001, 10001, &c. the multipliers for the different powers are,

1st power,	+ 1,	- 1,	+ 1,	- 1,	+ 1,	
2d power,	+ 1,	- 2,	+ 3,	- 4,	+ 5,	
3d power,	+ 1,	- 3,	+ 6,	- 10,	+ 15,	
4th power,	+ 1,	- 4,	+ 10,	- 20,	+ 35,	
5th power,	+ 1,	- 5,	+ 15,	- 35,	+ 70,	
6th power,	+ 1,	- 6,	+ 21,	- 56,	+ 126,	
7th power,	+ 1,	- 7,	+ 28,	- 84,	+ 210,	
8th power,	+ 1,	- 8,	+ 36,	- 120,	+ 330,	
9th power,	+ 1,	- 9,	+ 45,	- 165,	+ 495,	

The multipliers for other divisions are readily found: for example, let it be required to divide a given number by the 20th power of 11, 101, 1001, &c. The first multiplier will be + 1, the second - 20, the third $- 20 \times - 21 \div 2 = + 210$, the fourth $+ 210 \times - 22 \div 2 = - 1540$, and so on in other cases.

6. Required the value of $7854 \downarrow 3, \bar{2}, 2$, which is the same as saying, multiply 7854 by $(11)^3$ and then $(1001)^2$, and divide by $(101)^2$.

$$\begin{array}{r}
 \begin{array}{|c|c|c|c|c|c|c|}
 \hline
 7 & 8 & 5 & 4 & 0 & 0 & 0 \\
 \hline
 2 & 2 & 5 & 6 & 2 & 0 & 0 \\
 \hline
 2 & 2 & 5 & 6 & 2 & 0 & 0 \\
 \hline
 7 & 8 & 5 & 4 & & & \\
 \hline
 \end{array}
 & \downarrow 3, \\
 \\
 \begin{array}{|c|c|c|c|c|c|c|}
 \hline
 103 & 436 & 74 & & & & \\
 \hline
 206 & 87 & & & & & \\
 \hline
 & 10 & & & & & \\
 \hline
 \end{array}
 & \downarrow 3, 0, 2, \\
 \\
 \begin{array}{r}
 10364371 + \\
 207287 - \\
 3109 + \\
 41 - \\
 \hline
 10160152, \text{ the first eight figures.}
 \end{array}
 & \downarrow 3, \bar{2}, 2,
 \end{array}$$

7. Find the first seven figures of the quotient of 360 by 57.2957795 .

$$57.2957795 \downarrow 0, 0, 0, 7, 3, 6, 6, = 57.3$$

$$\therefore 57.3 \downarrow 0, 0, 0, \bar{7}, \bar{3}, \bar{6}, \bar{6}, = 57.2957795$$

$\therefore \frac{360, \downarrow 0,0,0,7,3,6,6}{57.3}$ gives the required quotient ;

$$\text{but } \frac{360}{57.3} = \frac{1200}{191} = 6.2827225$$

$$\begin{array}{r} \downarrow 0,0,0,7, \\ 3, \\ 6, \\ 6, \\ \hline 6.2827 \overline{) 225 \dots} \\ \underline{4398 \dots} \\ 189 \dots \\ \underline{37 \dots} \\ 4 \dots \end{array}$$

6.2831853, quotient.

8. Divide 70710678 by 39269908, and give the first seven figures of the quotient.

$$39269908 \downarrow 0,0,1,8,4,7,4,6 = 40000000 ;$$

then, according to the reasoning employed in the last example,

$$70710678 \div 4 = 17677669$$

$$\therefore 17677669 \downarrow 0,0,1,8,4,7,4,6 = 1.8006314, \text{ the quotient.}$$

9. Divide 3.1415927 by 1.4142136, and give the first seven figures of the quotient.

Divide 1.414 into 3.1415927 by common division, and the quotient is 2.2217770.

$$\begin{array}{r} \downarrow 0,0,0,1, \\ 5, \\ 1, \\ \hline 1.414 \overline{) 0000} \\ \underline{1414 \dots} \\ 707 \dots \\ \underline{4 \dots} \\ 1.4142125 \\ \hline 2.221 \overline{) 7770} \\ \underline{2222 \dots} \\ 1111 \dots \\ \underline{22 \dots} \\ 2.2214415, \text{ required quotient.} \end{array}$$

PART III.

EVOLUTION AND INVOLUTION OF DUAL
ARITHMETIC.

1. *Required the first eight figures of the cube of .4852.*

$$.48 \times .48 \times .48 = .110592$$

may be taken from a table of cubes, &c. or found by common multiplication.

$$48 \downarrow 0,1,0,8,2,4,7,8, = 4852.$$

$$(48)^3 \downarrow 0,3,0,24,6,12,21,24, = (4852)^3.$$

3	$(48)^3 = 11 \overline{) 059200}$ $ 33 \overline{) 1776}$ $ 33 \overline{) 18}$ $ 11$	$\downarrow 0,3,$
$34182915 \overline{) 27346 \dots} - 8$	$1139 \overline{) 4305} \dots$ $ 27346 \dots = A$ $ 32 \dots = A \times 23 \div 2$	
$\downarrow 0,3,0,24,$ $ 6,$ $ 12,$ $ 21,$ $ 24,$	$11421 \overline{) 683} \dots$ $ 685 \dots$ $ 137 \dots$ $ 24 \dots$ $ 3 \dots$	
	$11422532 = \text{cube of } .4852, \text{ true}$	

to the last figure.

2. *What is the fourth power of .88, true to eight places of figures?*

$$(.8)^3 = .4096$$

and

$$.8 \downarrow 1, = .88$$

$$\therefore (.8)^4 \downarrow 4, = (.88)^4$$

$$\begin{array}{r}
 (.8)^4 = \begin{array}{r}
 4 \overline{) 09600000} \\
 \underline{16} \\
 24 \\
 \underline{16} \\
 80 \\
 \underline{80} \\
 00 \\
 \underline{00} \\
 00 \\
 \underline{00} \\
 00
 \end{array} \downarrow 4,
 \end{array}$$

$$(.88)^4 = \underline{\underline{.59969536}}$$

3. *Required the first seven figures of the cube of .0176325.*

$(.017)^3 = .000004913$, which may be called .0000049, as the result is only required to seven places of decimals.

$$17 \downarrow 0,3,6,6,8, = 176325$$

$$\therefore (.017)^3 \downarrow 0,9,18,18,24, = (.0176324)^3.$$

$$\begin{array}{r}
 \begin{array}{r}
 3 \overline{) 17} - 3 \quad 49 \overline{) \dots} \quad \downarrow 0,9, \\
 4 \dots \overline{) \dots} \quad 4 \dots \overline{) \dots}
 \end{array} \\
 \begin{array}{r}
 169 \overline{) \dots} \quad 53 \overline{) \dots} \quad \downarrow 0,9,18, \\
 1 \dots \overline{) \dots} - 6 \quad 1 \overline{) \dots}
 \end{array} \\
 54
 \end{array}$$

$$\therefore .0000054 = \text{cube of } .0176325.$$

Another Method.

$$\begin{array}{r}
 49 \overline{) \dots} \downarrow 0,11, \\
 6 \overline{) \dots}
 \end{array}$$

$$\underline{\underline{.0000055}} = \text{cube of } .0176325.$$

Since one figure only is required,

$$(.017)^3 \downarrow 0,11, \text{ is put for } (.017)^3 \downarrow 0,9,18,18,24,$$

because, as is readily shown, $\downarrow 0,0,18$, is nearly equal $\downarrow 0,2$,

$$\begin{array}{r}
 100|000|000|000 \\
 1|000|000|000 \\
 45|00|000 \\
 12|000 \\
 21
 \end{array}
 \quad \downarrow 0,0,10,$$

$$\begin{array}{r}
 + 10100|45120|21\dots \\
 - \quad \quad |40401|80\dots \downarrow 0,0,10,0,\bar{4}, \\
 + \quad \quad \quad |101\dots
 \end{array}$$

$$\begin{array}{r}
 101000|471942 \\
 - \quad \quad |404002 \quad \downarrow 0,0,10,0,\bar{4},\bar{4}, \\
 + \quad \quad \quad |1
 \end{array}$$

$$\begin{array}{r}
 1010000|67941\dots \\
 - \quad \quad |60600\dots \\
 - \quad \quad |7070\dots \\
 - \quad \quad |202\dots \\
 - \quad \quad |70\dots
 \end{array}$$

$$\therefore \downarrow 0,0,10,0,\bar{4},\bar{4},\bar{6},\bar{7},\bar{2},\bar{7}, = \downarrow 0,1,$$

$$\therefore (1.001)^{10} \text{ is nearly equal } (1.01).$$

$$\begin{array}{r}
 1000|0000|0000 \\
 1|0000|0000 \\
 45|000 \\
 12
 \end{array}
 \quad \downarrow 0,0,10,$$

$$\begin{array}{r}
 + 1001000|45012\dots \\
 - \quad \quad |40040\dots \\
 - \quad \quad |4004\dots \\
 - \quad \quad |901\dots \\
 - \quad \quad |60\dots \\
 - \quad \quad |7\dots
 \end{array}$$

$$100100000000$$

$$\therefore \downarrow 0,0,0,10,0,0,\bar{4},\bar{4},\bar{9},\bar{6},\bar{7}, = \downarrow 0,0,1,$$

$$\therefore (1.0001)^{10} \text{ nearly equal } (1.001).$$

$$\begin{array}{r}
 10000|00000|00\dots \\
 1|00000|00\dots \downarrow 0,0,0,0,10, \\
 450\dots
 \end{array}$$

$$\begin{array}{r}
 + 100010000|450 \\
 - \quad \quad |400 \\
 - \quad \quad |50
 \end{array}$$

$$100010000000$$

$$\therefore \downarrow 0,0,0,0,10,0,0,0,0,\bar{4},\bar{5} = \downarrow 0,0,0,1,$$

$$\therefore (1.00001)^{10} \text{ nearly equal } (1.0001).$$

Working to seven places of decimals,

$$\downarrow 0,0,0,0,10,0,0 = \downarrow 0,0,0,1,$$

$$\downarrow 0,0,0,10,0,0,\bar{4} = \downarrow 0,0,1,$$

$$\downarrow 0,0,10,0,\bar{4},\bar{4},\bar{7} = \downarrow 0,1,$$

$$\downarrow 0,10,\bar{4},\bar{1},\bar{9},\bar{5},\bar{1} = \downarrow 1,$$

The following arrangement may be more convenient :—

$$\downarrow 1, \quad = \downarrow 0,10,\bar{4},\bar{1},\bar{9},\bar{5},\bar{1},$$

$$\downarrow 0,1, \quad = \downarrow 0,0,10,0,\bar{4},\bar{4},\bar{7},$$

$$\downarrow 0,0,1, \quad = \downarrow 0,0,0,10,0,0,\bar{4},$$

$$\downarrow 0,0,0,1, \quad = \downarrow 0,0,0,0,10,0,0, \quad (B).$$

$$\downarrow 0,0,0,0,1, \quad = \downarrow 0,0,0,0,0,10,0,$$

$$\downarrow 0,0,0,0,0,1, \quad = \downarrow 0,0,0,0,0,0,10,$$

$$\&c. \quad = \quad \&c.$$

To illustrate this property, take an example. It may be found by common arithmetic, that

$$(\cdot 0176325)^3 = \cdot 000005482033404328125.$$

For the purpose in view, twelve of these figures will suffice,

$$\underline{548203340433.}$$

$$\begin{array}{r}
 \underline{\underline{548203340430}} \\
 (17)^3 = 4913 \\
 \underline{4913} \\
 54 \overline{) 043000} \\
 \underline{54} \overline{) 0430} \\
 545 \overline{) 834300000} \\
 \underline{2183} \overline{) 337200} \\
 \underline{3275} \overline{) 006} \\
 \underline{2183} \\
 5480 \overline{) 20914389} \\
 \underline{16440} \overline{) 6274} \\
 \underline{16441} \\
 \underline{1} \\
 54818 \overline{) 5337105...} \\
 \underline{164455} \overline{) 60...} \\
 \underline{164...} \\
 548201 \overline{) 782829} \\
 \underline{1096404} \\
 \underline{1} \\
 548 \quad 028 \overline{) 79234...} \\
 \underline{43} \overline{) 8562...} \\
 \underline{21} \overline{) 928...} \\
 \underline{548} \\
 \underline{109} \\
 \underline{49}
 \end{array}
 \begin{array}{l}
 \downarrow 1, \\
 \downarrow 1,1, \\
 \downarrow 1,1,4, \\
 \downarrow 1,1,4,3, \\
 \downarrow 1,1,4,3,3, \\
 \downarrow 1,1,4,3,3,2, \\
 \begin{array}{l}
 8, \\
 4, \\
 1, \\
 2, \\
 9,
 \end{array}
 \end{array}$$

$$\therefore (017)^3 \downarrow 1,1,4,3,3,2,8,4,1,2,9, = \dots (0176325)^3. (A).$$

$$\begin{array}{r}
 0176325000000 \\
 \underline{17} \overline{) 00000000} \\
 \underline{51} \overline{) 00000000} \\
 \underline{51} \overline{) 000000} \\
 \underline{17} \overline{) 0000} \\
 175 \overline{) 151170000} \\
 \underline{1050907020} \\
 \underline{2627268} \\
 \underline{3503} \\
 \underline{3} \\
 176204707794
 \end{array}
 \begin{array}{l}
 \downarrow 0,3, \\
 \downarrow 0,3,6,
 \end{array}$$

$$\begin{array}{r}
 1762 \mid 0470 \mid 7794 \\
 1 \mid 0572 \mid 2824 \downarrow 0,3,6,6, \\
 \quad \quad 2 \mid 6531 \\
 \quad \quad \quad 4 \\
 \hline
 17631 \mid 04570 \mid 53 \dots \\
 1 \mid 41048 \mid 36 \dots \downarrow 0,3,6,6,8, \\
 \quad \quad \quad 494 \dots \\
 \hline
 176324562383 \\
 \quad \quad 3 \mid 52649 \dots \\
 \quad \quad 7 \mid 0530 \dots \\
 \quad \quad 1 \mid 4106 \dots \\
 \quad \quad \quad 1 \mid 76 \dots \\
 \quad \quad \quad 1 \mid 41 \dots \\
 \quad \quad \quad 1 \mid 4 \dots \\
 \hline
 176325000000
 \end{array}$$

$\downarrow 0,3,6,6,8,$
 $2,$
 $4,$
 $8,$
 $1,$
 $8,$

$$\therefore (017)^3 \downarrow 0,9,18,18,24,6,12,24,3,24,24, = (0176325)^3. \quad (D).$$

(A) being equal (D),

$$\therefore \downarrow 1,1,4,3,3,2,8,4,1,2,9, = \downarrow 0,9,18,18,24,6,12,24,3,24,24,$$

Although these expressions appear unequal, they, as factors, effect the same purpose; each of the indices on the left is less than 10, while some of the indices on the right amount to 18 and 24.

The equality of such multipliers is at once established by equations (B), for, if

$\downarrow 0,0,0,20,0,0,8,$ be subtracted, $\downarrow 0,0,2,$ may be added without disturbing the equality. In the same way, if

$\downarrow 0,0,20,0,8,8,14,$ be subtracted, $\downarrow 0,1,$ must be added to maintain the equality. It is easily shown that the following six expressions are equal to one another:—

This example has not for its object the extracting of the cube root of a number, to effect which, as I will presently show, is an easy operation by this system of Arithmetic.

My design here is to illustrate a principle. The process is extremely simple, and requires little mental labour, yet it takes considerable space to explain the matter fully. However, I prefer prolixity to obscurity, ordinary examples to wonderful developments, lengthy work to contracted operations. When the method is understood, the operator may contract as much as he pleases; develop all that his fancy dictates by the most dark and difficult symbols, never to be read or understood by any one after him.

4. *Required the cube root of 577385.261, true to nine places of figures.*

$$(80)^3 = 512000.$$

$$\begin{array}{r} 3 \text{ times} \\ 153600000 \mid 0 \\ 61440000 \mid \sim 4 \end{array}$$

↓ 0,12,

$$\begin{array}{r} 3 \text{ times} \\ 17308 \mid 0324 \mid 5 \\ 34616 \mid 1 \dots \mid \sim 2 \end{array} \quad \downarrow 0,12,0,6,$$

$$\begin{array}{r} 3 \text{ times} \\ 173184 \mid 1989 \\ 103911 \mid \sim 6 \end{array} \quad \downarrow 0,12,0,6,18,$$

$$\begin{array}{r} 3 \text{ times} \\ 1732 \dots \mid \\ 520 \dots \mid \sim 3 \end{array}$$

$$\begin{array}{r} 3 \text{ times} \\ 1732 \dots \mid \\ 155 \dots \mid \sim 9 \end{array}$$

$$\begin{array}{r} * \\ 51 \mid 20 \mid 00 \mid 00 \mid 0 \dots \sim 1 \\ 6 \mid 14 \mid 40 \mid 00 \mid 0 \dots \sim 12 = A \\ 33 \mid 79 \mid 20 \mid 0 \dots \sim 11 \times A \div 2 = B \\ 1 \mid 12 \mid 64 \mid 0 \dots \sim 10 \times B \div 3 = C \\ 2 \mid 53 \mid 4 \dots \sim 9 \times C \div 4 = D \\ 4 \mid 1 \dots \sim 8 \times D \div 5 = E \end{array}$$

$$\begin{array}{r} 5769 \mid 3441 \mid 5 \dots \\ 34616 \mid 1 \dots \\ 87 \dots \end{array}$$

$$\begin{array}{r} 57728 \mid 0663 \dots \sim 1 \\ 103911 \dots \sim 18 = A \\ 9 \dots \sim 17 \times A \div 2 = B \end{array}$$

$$\begin{array}{r} 5773845 \mid 83 \dots \\ 52 \mid 0 \dots \end{array}$$

$$15 \mid 5 \dots$$

$$577385258$$

$$512 \downarrow 0,12,0,6,18,0,9,27, = 577385.258;$$

the cube root of which will be represented by

$$8 \downarrow 0,4,0,2,6,0,3,9,$$

$\downarrow 0,4,$

$$\begin{array}{r} 80 \overline{) 00 \overline{) 00 \overline{) 00 \overline{) 0}} \\ 3 \overline{) 20 \overline{) 00 \overline{) 00 \overline{) 0}} \\ 4 \overline{) 80 \overline{) 00 \overline{) 0}} \\ 3 \overline{) 20 \overline{) 0}} \\ 8 \end{array}$$

$\downarrow 0,4,0,2,$

$$\begin{array}{r} 83 \overline{) 24 \overline{) 83 \overline{) 20 \overline{) 8}} \dots \\ 1 \overline{) 66 \overline{) 49 \overline{) 7}} \dots \\ 8 \end{array}$$

$\downarrow 0,4,0,2,6,$

$$\begin{array}{r} 83 \overline{) 26 \overline{) 49 \overline{) 71 \overline{) 3}} \dots \\ 4 \overline{) 99 \overline{) 59}} \\ 1 \end{array}$$

$0,3,$

$9,$

$$\begin{array}{r} 83 \overline{) 26 \overline{) 99 \overline{) 67 \overline{) 3}} \dots \dots \dots \\ 2 \overline{) 5 \overline{) 0}} \dots \dots \dots \\ 7 \overline{) 5} \dots \dots \dots \end{array}$$

$\therefore 83.2699998$ is the cube root of 577385.261 .

5. Required the first six figures of the fourth root of 3527.63 .

$$7^4 = 2401.$$

$$\begin{array}{r} 4 \text{ times} \\ 96040 \overline{) 0} \\ 96040 \overline{) \sim 1} \end{array}$$

$$\begin{array}{r} 3 \overline{) 5 \overline{) 2 \overline{) 7 \overline{) 6 \overline{) 3}} \\ 2 \overline{) 4 \overline{) 0 \overline{) 1 \overline{) 0 \overline{) 0}} \\ 9 \overline{) 6 \overline{) 0 \overline{) 4 \overline{) 0}} \\ 1 \overline{) 4 \overline{) 4 \overline{) 0 \overline{) 6}} \\ 9 \overline{) 6 \overline{) 0}} \\ 2 \overline{) 4} \end{array}$$

$\downarrow 4,$

$$\begin{array}{r} 4 \text{ times} \\ 14061 \overline{) 20} \\ 1125 \dots \overline{) \sim 8} \end{array}$$

$$\begin{array}{r} 3 \overline{) 5 \overline{) 1 \overline{) 5 \overline{) 3 \overline{) 0}} \dots \dots \\ 1 \overline{) 1 \overline{) 2 \overline{) 5}} \dots \dots \\ 2 \dots \end{array}$$

$\downarrow 4,0,0,32,$

$$\begin{array}{r} 4 \text{ times} \\ 141062 \overline{) 8} \\ 99 \dots \overline{) \sim 7} \end{array}$$

$$\begin{array}{r} 3 \overline{) 5 \overline{) 2 \overline{) 6 \overline{) 5 \overline{) 7}} \dots \dots \dots \\ 9 \overline{) 9 \overline{) \dots \dots \dots}} \\ 7 \overline{) \dots \dots \dots} \end{array}$$

28,
20,

Given number $3 \overline{) 5 \overline{) 2 \overline{) 7 \overline{) 6 \overline{) 3}}}$

The fourth root of $2401 \downarrow 4,0,0,32,28,20, = 7 \downarrow 1,0,0,8,7,5,$

$$\begin{array}{r}
 700000 \\
 \underline{70000} \quad \downarrow 1, \\
 7700|00 \dots \\
 \quad 62|6 \dots \dots 0,0,8, \\
 \quad \quad 54| \dots \dots \quad 7, \\
 \quad \quad \quad 4| \dots \dots \quad \quad 5, \\
 \hline
 770684
 \end{array}$$

$\therefore 7.70684 = \text{fourth root of } 3527.63.$

6. Required the first six figures of the sixth root of 3856700.

$$20^5 = 3200000.$$

5 times	38 5 6 7 0 0
1600000 0	32 00 00 0 . I
48000 .. ~ 3	4 80 00 . . 15 = A
	33 60 . . 14 × A ÷ 2 = C
	1 46 . . 13 × C ÷ 3 = D
	4 . . 12 × D ÷ 4 = E
5 times	37 1 5 1 0 I
1857 550	13 003 35 = A
13003 ~ 7	221 34 × A ÷ 2 = B
	2 33 × B ÷ 3 = C
5 times	38 4 7 3 6 . . I
19236 80	7 69 . . 20
769 .. ~ 4	I . . 19 × A ÷ 2
5 times	38 5 5 0 6 . . .
193 ...	I 5 4
154 .. ~ 8	
5 times	
193	
9 .. ~ 5	I 0
	38 5 6 7 0

$\therefore 20 \downarrow 0,3,7,4,8,5,$ expresses the required fifth root.

$$\begin{array}{r} 20 \overline{) 00 \overline{) 00}} \\ \underline{60} \overline{) 00} \\ 60 \end{array} \quad \downarrow 0, 3,$$

$$\begin{array}{r} 206 \overline{) 060} \\ \underline{144} \overline{) 2} \\ 4 \end{array} \quad \downarrow 0, 3, 7,$$

$$\begin{array}{r} 2075 \overline{) 06 \dots} \\ \underline{83} \dots \quad 4, \\ \underline{17} \dots \quad 8, \\ \underline{1} \dots \quad 5, \end{array}$$

$\therefore 207607 = \text{the fifth root of } 3856700.$

7. Required the cube root of $\cdot 5236 = \sqrt[3]{\frac{\pi}{6}}$, the side of a cube equal the solidity of a sphere whose diameter = 1.

$$(\cdot 8)^3 = \cdot 512$$

$$\begin{array}{r} \begin{array}{l} 3 \text{ times} \\ 1536 \overline{) 00} \\ 1075 \cdot \quad \sim 7 \end{array} \quad \begin{array}{l} 3 \text{ times} \\ 157 \cdot \cdot \overline{) } \\ 63 \dots \quad \sim 4 \end{array} \quad \begin{array}{l} 3 \text{ times} \\ 157 \dots \overline{) } \\ 11 \dots \quad \sim 7 \end{array} \end{array} \quad \begin{array}{r} \cdot 52360 \\ \underline{512} \overline{) 00 \cdot \sim 1} \\ 10 \overline{) 75 \cdot \sim 21} = A \\ \underline{11} \cdot \sim 20 \times A \div 2 \\ \hline 5228 \overline{) 6 \dots} \\ \underline{63} \dots \\ \hline 52349 \overline{) \dots} \\ \underline{11} \dots \\ \hline 52360 \end{array}$$

$\therefore (\cdot 8) \downarrow 0, 0, 7, 4, 7$, expresses the required number;

$$\begin{array}{r} 800 \overline{) 00 \cdot} \\ \underline{560 \cdot} \\ 0 \quad 2 \cdot \end{array} \quad \downarrow 0, 0, 7,$$

$$\begin{array}{r} 8056 \overline{) 2 \dots} \\ \underline{32} \dots \quad 4, \\ \underline{6} \dots \quad 7, \end{array}$$

$\therefore (\cdot 5236)^{\frac{1}{3}} = \cdot 80600$ nearly.

8. Let it be required to find the $\frac{17}{43}$ power of $3\cdot141593$ true to five places of decimals.

$$1 \downarrow 12,0,1,0,0,8,6 = 3\cdot141593.$$

It is necessary to have 1, to the left of $\downarrow$, because $\frac{17}{43}$ power of 1 = 1. One or two trials show that $1 \downarrow 11$, is too small, and $1 \downarrow 13$, exceeds $3\cdot141593$.

$\downarrow 12,$	$\begin{array}{r} 1000000 \\ 1200000 \\ 660000 \\ 220000 \\ 49500 \\ 7920 \\ 924 \\ 79 \\ 5 \end{array}$	$\begin{array}{l} 1 \\ 12 \\ 11 \times A \div 2 = B \\ 10 \times B \div 3 = C \\ 9 \times C \div 4 = D \\ 8 \times D \div 5 = E \\ 7 \times E \div 6 = F \\ 6 \times F \div 7 = G \\ 5 \times G \div 8 = H \end{array}$
$\downarrow 12,0,1,$	$\begin{array}{r} 3138428 \dots \\ 3138 \dots \end{array}$	
$0,0,8,$ $6,$	$\begin{array}{r} 3141566 \dots \\ 25 \dots \\ 2 \dots \end{array}$	
$\therefore 1 \downarrow 12,0,1,0,0,8,6, = \underline{3141593}$		

The 17th power of $1 \downarrow 12,0,1,0,0,8,6, = 1 \downarrow 204,0,17,0,0,136,102,$ which has to be divided by 43 for the required result.

By means of expressions (B) all the indices to the right of $\downarrow$, in the last expression, may be reduced until they are divisible by 43.

43 into 204, goes 4 times and 32 over,

now, $\downarrow 32, = \downarrow 0,320,\overline{128},\overline{32},\overline{288},\overline{160},\overline{32},$

$$\begin{aligned}
 \therefore (B) \text{I} \downarrow 204,0,17,0,0,136,102 &= \text{I} \downarrow 172,320, \overline{111}, \overline{32,288}, \overline{24}, \overline{70}, \\
 &= \text{I} \downarrow 172,301, \overline{79}, \overline{32,364,100}, \overline{63}, \\
 &= \text{I} \downarrow 172,301, \overline{43,328,364,100,207}, \\
 &= \text{I} \downarrow 172,301, \overline{43,301}, \overline{94,100,207}, \\
 &= \text{I} \downarrow 172,301, \overline{43,301,106}, \overline{0}, \overline{7}, \\
 &= \text{I} \downarrow 172,301, \overline{43,301}, \overline{86,172,258},
 \end{aligned}$$

All the indices can, in this reduced form, be divided by 43. Consequently, the $\frac{17}{43}$ power, or root of 3.141593 , is represented by

$$\text{I} \downarrow 4,7,1,7,2,4,6,$$

$$\begin{array}{r}
 \text{I} \downarrow 0000000 \\
 \quad 4000000 \\
 \quad \quad 6000000 \\
 \quad \quad \quad 4000000 \\
 \quad \quad \quad \quad 1000000
 \end{array}$$

$$\begin{array}{r}
 \text{I} \downarrow 4641000. \\
 \quad \text{I} \downarrow 02487. \\
 \quad \quad 3074. \\
 \quad \quad \quad 51.
 \end{array}$$

$$\begin{array}{r}
 \text{I} \downarrow 569712. \\
 \quad \text{I} \downarrow 570.
 \end{array}$$

$$\begin{array}{r}
 \text{I} \downarrow 571282. \\
 \quad \text{I} \downarrow 1100.
 \end{array}$$

$$\begin{array}{r}
 \text{I} \downarrow 572382 \\
 \quad \quad 32 \text{ minus.} \\
 \quad \quad \quad 6 \text{ minus.}
 \end{array}$$

$$\text{I} \downarrow 572344$$

$$\therefore \text{I} \downarrow 572344 = (3.141593)^{\frac{17}{43}}.$$

9. Express $\downarrow 0,1$, in positive terms of the indices that follow in succession.

$$\begin{array}{r}
 \begin{array}{r|rrrr}
 100 & 000 & 000 & 000 \\
 & 900 & 000 & 000 \\
 & 360 & 000 & \\
 & 84 & 00 & \\
 & 13 & &
 \end{array}
 & \downarrow 0,0,9, \\
 \hline
 \begin{array}{r|rrrr}
 1009 & 0360 & 8413 \\
 & 9081 & 3247 \\
 & 36325 & \\
 & 8 &
 \end{array}
 & \downarrow 0,0,9,9, \\
 \hline
 \begin{array}{r|rrrr}
 10099 & 44579 & 93 \dots \\
 & 50497 & 23 \dots \\
 & 101 & \dots
 \end{array}
 & \\
 \hline
 \begin{array}{r|rrrr}
 100999 & 507817 \\
 & 403998 \\
 & 80800 \\
 & 7070 \\
 & 303 \\
 & 10
 \end{array}
 &
 \end{array}$$

$\downarrow 0,0,9,9,5,4,8,7,3,1,$
 $\therefore \downarrow 01, = \downarrow 0,0,9,9,5,4,8,7,3,1,$

The following equalities, found in the same manner, may be made useful in reducing one set of indices to another:

$$\begin{array}{ll}
 \downarrow 1, & = \downarrow 0,9,5,7,5,9,7,3,5,7,1,2,5,9,4,7,7, \\
 \downarrow 0,1, & = \downarrow 0,0,9,9,5,4,8,7,3,1,0,4,4,5,5,1,9, \\
 \downarrow 0,0,1, & = \downarrow 0,0,0,9,9,9,5,4,5,7,8,4,6,0,5,7,3, \\
 \downarrow 0,0,0,1, & = \downarrow 0,0,0,0,9,9,9,9,5,4,5,4,8,7,5,7,5, \\
 \downarrow 0,0,0,0,1, & = \downarrow 0,0,0,0,0,9,9,9,9,9,5,4,5,4,5,8,4, \\
 \downarrow 0,0,0,0,0,1, & = \downarrow 0,0,0,0,0,0,9,9,9,9,9,9,5,4,5,4,5, \\
 \downarrow 0,0,0,0,0,0,1, & = \downarrow 0,0,0,0,0,0,0,9,9,9,9,9,9,9,5,4,5, \\
 \downarrow 0,0,0,0,0,0,0,1, & = \downarrow 0,0,0,0,0,0,0,0,9,9,9,9,9,9,9,9,5, \\
 \&c. & = \&c.
 \end{array}$$

10. What power must 1.251853 be raised to, so that the result may be 1.571653?

$$\text{Or, } (1.251853)^x = 1.571653, \text{ find } x.$$

It is readily found that $1 \downarrow 2,3,4,1,5,6,0 = 1.251853$,
and that $1 \downarrow 4,7,1,2,3,6,0 = 1.571653$.

The co-efficients 1, to the left of $\downarrow$, in future operations will be omitted, $1 \downarrow 2,3,4,1,5,6,0$, is the same as $\downarrow 2,3,4,1,5,6,0$,

In the succeeding reduction, the following equalities, before given, are employed:—

$$\downarrow 1, = \downarrow 0, 10, \bar{4}, \bar{1}, \bar{9}, \bar{5}, \bar{1},$$

$$\downarrow 0,1, = \downarrow 0, 0, 10, 0, \bar{4}, \bar{4}, \bar{7}, (K).$$

$$\downarrow 0,0,1, = \downarrow 0, 0, 0, 10, 0, 0, \bar{4},$$

$$\begin{aligned} \downarrow 2,3,4,1,5,6,0, &= \downarrow 0, 23, \bar{4}, \bar{1}, \bar{13}, \bar{4}, \bar{2}, \\ &= \downarrow 0, 0, 226, \bar{1}, \bar{105}, \bar{96}, \bar{163}, \\ &= \downarrow 0, 0, 0, 2259, \bar{105}, \bar{96}, \bar{1067}, \end{aligned} \left. \vphantom{\begin{aligned} \downarrow 2,3,4,1,5,6,0, \\ &= \downarrow 0, 0, 226, \bar{1}, \bar{105}, \bar{96}, \bar{163}, \\ &= \downarrow 0, 0, 0, 2259, \bar{105}, \bar{96}, \bar{1067}, \end{aligned}} \right\}$$

22590

105

224850

96

2247540

1067

2246473

With (K) this reduction requires no mental labour, for when $\downarrow 2$, is taken away, and 0, put in its place, $\downarrow 0,20,\bar{8},\bar{2},\bar{18},\bar{10},\bar{2}$, is added.

$$0, 3, 4, 1, 5, 6, 0,$$

$$0, 20, \bar{8}, \bar{2}, \bar{18}, \bar{10}, \bar{2},$$

$$\downarrow 0, 23, \bar{4}, \bar{1}, \bar{13}, \bar{4}, \bar{2},$$

$$0, 0, 230, 0, \bar{92}, \bar{92}, \bar{161}, = 23 \text{ times } 0,0,10,0,4,4,7,$$

$$0, 0, 226, \bar{1}, \bar{105}, \bar{96}, \bar{163},$$

It will be found unnecessary to set down these figures to effect this continued reduction.

$$\begin{array}{r}
 \downarrow 4, \quad 7, \quad 1, \quad 2, \quad 3, \quad 6, \quad 0, \\
 = \downarrow 0, \quad 47, \quad \overline{15}, \quad \overline{2}, \quad \overline{33}, \quad \overline{14}, \quad \overline{4}, \\
 \downarrow 0, \quad 0, \quad 455, \quad \overline{2}, \quad \overline{221}, \quad \overline{202}, \quad \overline{333}, \\
 \downarrow 0, \quad 0, \quad 0, \quad 4548, \quad \overline{221}, \quad \overline{202}, \quad \overline{2153},
 \end{array}$$

$$\begin{array}{r}
 45480 \\
 221 \\
 \hline
 452590 \\
 202 \\
 \hline
 4523880 \\
 2153 \\
 \hline
 4521727
 \end{array}$$

$\therefore 4521727$ divided by 2246473 gives 2.012812 the value of x . Consequently 1.251853 raised to the 2.012812 power produces 1.571653 .

11. *What power must 10 be raised to, so that the result may be 7? Or in other terms, given $10^x = 7$, find x .*

It is easily shown that $1 \downarrow 24, 1, 5, 1, 9, 2, 5, 9 = 10$,
and that $1 \downarrow 20, 3, 9, 8, 6, 0, 1, 2 = 7$.

The following reduction is readily made by employing the equalities (K) :—

$$\begin{array}{r}
 10 = \downarrow 24, \quad 1, \quad 5, \quad 1, \quad 9, \quad 2, \quad 9, \\
 = 0, \quad 241, \quad \overline{91}, \quad \overline{23}, \quad \overline{207}, \quad \overline{118}, \quad \overline{15}, \\
 = 0, \quad 0, \quad 2319, \quad \overline{23}, \quad \overline{1171}, \quad \overline{1082}, \quad \overline{1702}, \\
 = 0, \quad 0, \quad 0, \quad 23167, \quad \overline{1171}, \quad \overline{1082}, \quad \overline{10978},
 \end{array}$$

$$\begin{array}{r}
 231670 \\
 1171 \\
 \hline
 2304990 \\
 1082 \\
 \hline
 23039080 \\
 10978 \\
 \hline
 23028102
 \end{array}$$

$$\begin{array}{r}
 7 = \downarrow 20, \quad 3, \quad 9, \quad 8, \quad 6, \quad 0, \quad 1, \\
 \quad 0, \quad 203, \quad \overline{71}, \quad \overline{12}, \quad \overline{174}, \quad \overline{100}, \quad \overline{19}, \\
 \quad 0, \quad 0, \quad 1959, \quad \overline{12}, \quad \overline{986}, \quad \overline{912}, \quad \overline{1440}, \\
 \quad 0, \quad 0, \quad 0, \quad 19578, \quad \overline{986}, \quad \overline{912}, \quad \overline{9276},
 \end{array}$$

$$\begin{array}{r}
 195780 \\
 \underline{986}
 \end{array}$$

$$\begin{array}{r}
 1947940 \\
 \underline{912}
 \end{array}$$

$$\begin{array}{r}
 19470280 \\
 \underline{9276}
 \end{array}$$

$$\begin{array}{r}
 19461004
 \end{array}$$

$\therefore$ 19461004 divided by 23028102 gives .8450980, the logarithm of 7 true to the last figure.

If the logarithm be only required to five places of figures, but five factors are necessary.

$$\begin{array}{r}
 10 = \downarrow 24, \quad 1, \quad 5, \quad 1, \quad 9, \\
 \quad 0, \quad 241, \quad \overline{91}, \quad \overline{23}, \quad \overline{207}, \\
 \quad 0, \quad 0, \quad 2319, \quad \overline{23}, \quad \overline{1171},
 \end{array}$$

$$\begin{array}{r}
 23190 \\
 \underline{23}
 \end{array}$$

$$\begin{array}{r}
 231670 \\
 \underline{1171}
 \end{array}$$

$$\begin{array}{r}
 230499
 \end{array}$$

$$\begin{array}{r}
 7 = \downarrow 20, \quad 3, \quad 9, \quad 8, \quad 6, \\
 \quad 0, \quad 203, \quad \overline{71}, \quad \overline{12}, \quad \overline{174}, \\
 \quad 0, \quad 0, \quad \overline{1959}, \quad \overline{12}, \quad \overline{986},
 \end{array}$$

$$\begin{array}{r}
 19590 \\
 \underline{12}
 \end{array}$$

$$\begin{array}{r}
 195780 \\
 \underline{986}
 \end{array}$$

$$\begin{array}{r}
 194794
 \end{array}$$

194794 divided by 230499 gives $\cdot 845099$, the logarithm of 7, to six decimal places, and differs but a unit from the truth; the logarithm of 7 taken from a table is $\cdot 845098$.

12. *Find in a direct manner the logarithm of 8.*

$$\frac{10}{8} = 1\cdot 25 \text{ and } 1 \downarrow 2,3,2,6,7,3,2, = 1\cdot 25.$$

But $\downarrow 2,3,2,6,7,3,2$, by (K) may be reduced to 2231653. In the last problem it was found that $10 = \downarrow 24,1,5,1,9,2,9, = 23028102$, then 2231653 divided by 23028080, by common division, gives $\cdot 09690999$, the logarithm of $1\cdot 25$.

$\log 10 = 1\cdot 00000000$
 $\log 1\cdot 25 = \cdot 0969100$
 $\log \text{ of } 8 = \cdot 9030900$ true to seven places
of decimals. To seven places of decimals $\cdot 09690999$ is represented by $\cdot 0969100$.

The cube root of $8 = 2$,

$$\therefore \log 2 = \cdot 3010300.$$

$$\begin{aligned} 1\cdot 1 &= 1 \downarrow 1, & 0, & 0, & 0, & 0, & 0, \\ (K) &= \downarrow 0, & 10, & \bar{4}, & \bar{1}, & \bar{9}, & \bar{5}, & \bar{1}, \\ &= \downarrow 0, & 0, & 96, & \bar{1}, & \bar{49}, & \bar{45}, & \bar{71}, \\ &= \downarrow 0, & 0, & 0, & 959, & \bar{49}, & \bar{45}, & \bar{455}, \end{aligned}$$

$$\begin{array}{r} 9590 \\ 49 \\ \hline 95410 \\ 45 \\ \hline 953650 \\ 455 \\ \hline \end{array}$$

953195 divided by 23028102 gives $\cdot 0413927$, the logarithm of $1\cdot 1$; and therefore the logarithm of $11 = 1\cdot 0413927$.

The reason of this rule is evident, for $\downarrow 2,3,5,4,8,2,4$, from the properties (K), may be reduced to the form $\downarrow 0,0,0,0,0,0, N$,; and any other expression, as $\downarrow 1,3,6,2,5,7,2$, may, in the same way, be represented by $\downarrow 0,0,0,0,0,0, M$,:

N is an index of the factor 10000001, and M is an index of the same factor; any other factor may be employed.

Let a = the extreme factor; in the case before us $a = 10000001$. Let it be required to find the power $\downarrow 2,3,5,4,8,2,4$, must be raised to, to produce $\downarrow 1,3,6,2,5,7,2$; if x = the required power, then

$$(\downarrow 0,0,0,0,0,0, N)^x = \downarrow 0,0,0,0,0,0, M,$$

$$\therefore (a^N)^x = a^M$$

$$\therefore x \log (a^N) = M \log a,$$

$$\therefore N x \log a = M \log a,$$

$$\therefore x = \frac{M}{N}$$

13. *The reciprocal of .743383 is 1.3452, what are the logarithms of both these numbers?*

$$1.34520 = \downarrow 3, \quad 1, \quad 0, \quad 6, \quad 6, \quad 0,$$

$$= \downarrow 0, \quad 31, \quad \overline{12}, \quad \overline{3}, \quad \overline{21}, \quad \overline{15},$$

$$= \downarrow 0, \quad 0, \quad 298, \quad 3, \quad \overline{145}, \quad \overline{170},$$

The equalities (K) are only employed to six figures. $\left\{ \begin{array}{l} \downarrow 1, \quad = \downarrow 0, \quad 10, \quad \overline{4}, \quad \overline{1}, \quad \overline{9}, \quad \overline{5}, \\ \downarrow 0,1, \quad = \downarrow 0, \quad 0, \quad 10, \quad 0, \quad \overline{4}, \quad \overline{5}, \\ \downarrow 0,0,1, \quad = \downarrow 0, \quad 0, \quad 0, \quad 10, \quad 0, \quad 0, \end{array} \right.$

$$\begin{array}{r} 29830 \\ \underline{145} \\ 296850 \\ \underline{170} \\ 296680 \end{array}$$

296680 divided by 2302810, the constant before used, gives
 ·128786, the logarithm of 1·3452.

$$\begin{array}{r} \log 10 = 1\cdot000000 \\ \log 1\cdot3452 = \cdot128786 \\ \hline \cdot871214 = \log \text{ of } 743383. \end{array}$$

10, is the base of the common system of logarithms, and
 2·718281828459, is the base of the hyperbolic system, which is
 by most writers represented by ϵ .

14. *Required the hyperbolic logarithm of $\pi = 3\cdot14159265359$,
 true to seven places of decimals.*

$$\epsilon^x = \pi$$

is the equation to be solved.

$$\pi \text{ is found} = \downarrow 12, 0, 1, 0, 0, 8, 2, 3,$$

$$\epsilon \quad \text{,,} \quad = \downarrow 10, 4, 7, 1, 0, 0, 3, 8,$$

$$\begin{array}{r} \downarrow 12, \quad 0, \quad 1, \quad 0, \quad 0, \quad 8, \quad 2, \\ = \downarrow 0, \quad 120, \quad \overline{47}, \quad \overline{12}, \quad \overline{108}, \quad \overline{52}, \quad \overline{10}, \\ = \downarrow 0, \quad 0, \quad 1153, \quad \overline{12}, \quad \overline{588}, \quad \overline{532}, \quad \overline{850}, \\ = \downarrow 0, \quad 0, \quad 0, \quad 11518, \quad \overline{588}, \quad \overline{532}, \quad \overline{5462}, \end{array}$$

$$\begin{array}{r} 115180 \\ 588 \\ \hline \end{array}$$

$$\begin{array}{r} 1145920 \\ 532 \\ \hline \end{array}$$

$$\begin{array}{r} 11453880 \\ 5462 \\ \hline \end{array}$$

$$11448418$$

$$\begin{array}{r}
 \downarrow 10, \quad 4, \quad 7, \quad 1, \quad 0, \quad 0, \quad 4, \\
 = \downarrow 0, \quad 104, \quad \overline{33}, \quad \overline{9}, \quad \overline{90}, \quad \overline{50}, \quad \overline{6}, \\
 \downarrow 0, \quad 0, \quad 1007, \quad \overline{9}, \quad \overline{506}, \quad \overline{466}, \quad \overline{734}, \\
 \downarrow 0, \quad 0, \quad 0, \quad 10061, \quad \overline{506}, \quad \overline{466}, \quad \overline{4762},
 \end{array}$$

$$\begin{array}{r}
 100610 \\
 \quad 506 \\
 \hline
 1001040 \\
 \quad 466 \\
 \hline
 10005740 \\
 \quad 4762 \\
 \hline
 10000978
 \end{array}$$

Then 11448418 divided by 10000978 by common division gives 1.1447297, the hyperbolic logarithm of π . This example may be more readily solved by taking the square roots of π and ϵ , for the determination of $\downarrow 12$, and $\downarrow 10$, requires more skill than the finding of $\downarrow 6$, and $\downarrow 5$; the fourth, eighth, or any other convenient roots of π and ϵ , may be operated with in the same manner.

$$\sqrt{\epsilon} = 1.6487213 = \downarrow 5, 2, 3, 5, 5, 0, 0,$$

$$\sqrt{\pi} = 1.77245385 = \downarrow 6, 0, 0, 5, 0, 3, 9,$$

$$\begin{array}{r}
 \downarrow 5, \quad 2, \quad 3, \quad 5, \quad 5, \quad 0, \quad 0, \\
 = \downarrow 0, \quad 52, \quad \overline{17}, \quad 0, \quad \overline{40}, \quad \overline{25}, \quad \overline{5}, \\
 = \downarrow 0, \quad 0, \quad 503, \quad 0, \quad \overline{248}, \quad \overline{233}, \quad \overline{369}, \\
 = \downarrow 0, \quad 0, \quad 0, \quad 5030, \quad \overline{248}, \quad \overline{233}, \quad \overline{2884},
 \end{array}$$

$$\begin{array}{r}
 50300 \\
 \quad 248 \\
 \hline
 500520 \\
 \quad 233 \\
 \hline
 5002870 \\
 \quad 2381 \\
 \hline
 5000489
 \end{array}$$

$$\begin{aligned}
& \downarrow 6, \quad 0, \quad 0, \quad 5, \quad 0, \quad 3, \quad 9, \\
& = \downarrow 0, \quad 60, \quad \overline{24}, \quad \overline{1}, \quad \overline{54}, \quad \overline{27}, \quad \overline{3}, \\
& = \downarrow 0, \quad 0, \quad 576, \quad \overline{1}, \quad \overline{294}, \quad \overline{267}, \quad \overline{417}, \\
& = \downarrow 0, \quad 0, \quad 0, \quad 5759, \quad \overline{294}, \quad \overline{267}, \quad \overline{2721},
\end{aligned}$$

$$\begin{array}{r}
57590 \\
\underline{294} \\
572960 \\
\underline{267} \\
5726930 \\
\underline{2721} \\
5724209
\end{array}$$

Divide 5000489 into 5724209 by common division, the quotient = 1.1447298, the hyperbolic logarithm of π , true to the last figure: the equalities used in this latter method are,

$$\begin{aligned}
\downarrow 1, & \quad = \downarrow 0, \quad 10, \quad \overline{4}, \quad \overline{1}, \quad \overline{9}, \quad \overline{5}, \quad \overline{1}, \\
\downarrow 0, 1, & \quad = \downarrow 0, \quad 0, \quad 10, \quad 0, \quad \overline{4}, \quad \overline{4}, \quad \overline{7}, \\
\downarrow 0, 0, 1, & \quad = \downarrow 0, \quad 0, \quad 0, \quad 10, \quad 0, \quad 0, \quad \overline{5}, \quad (K). \\
\downarrow 0, 0, 0, 1, & \quad = \downarrow 0, \quad 0, \quad 0, \quad 0, \quad 10, \quad 0, \quad 0,
\end{aligned}$$

The following equations of condition are composed of positive number, and may be often employed with advantage, when only seven decimal places are required:—

$$\begin{aligned}
\downarrow 1, & \quad = \downarrow 0, \quad 9, \quad 5, \quad 7, \quad 5, \quad 9, \quad 7, \\
\downarrow 0, 1, & \quad = \downarrow 0, \quad 0, \quad 9, \quad 9, \quad 5, \quad 4, \quad 9, \\
\downarrow 0, 0, 1, & \quad = \downarrow 0, \quad 0, \quad 0, \quad 9, \quad 9, \quad 9, \quad 5, \quad (L). \\
\downarrow 0, 0, 0, 1, & \quad = \downarrow 0, \quad 0, \quad 0, \quad 0, \quad 10, \quad 0, \quad 0,
\end{aligned}$$

Example.

Reduce $\epsilon = \downarrow 10, 4, 7, 1, 0, 0, 4$, to a representative number, standing in the seventh position by the equations (L).

$$\downarrow 10, = \downarrow 0, 90, 50, 70, 50, 90, 70,$$

subtract the left-hand member of this equation, and add the right-hand member, the result will be —

$$\epsilon = \downarrow 0, 94, 57, 71, 50, 90, 74,$$

Again, $\downarrow 0, 94 = \downarrow 0, 0, 846, 846, 470, 376, 846,$

take the left-hand member away, and add the right-hand member ;

$$\begin{aligned} \text{Then } \epsilon &= \downarrow 0, 0, 903, 917, 520, 466, 920, \\ \downarrow 0, 0, 903, &= \downarrow 0, 0, 0, 8127, 8127, 8127, 4515, \\ \therefore \epsilon &= \downarrow 0, 0, 0, 9044, 8647, 8593, 5435, \\ \therefore \epsilon &= \downarrow 0, 0, 0, 0, 0, 0, 10000065, \end{aligned}$$

$$\begin{array}{r} 90440 \\ 8647 \\ \hline 990870 \\ 8593 \\ \hline 9994630 \\ 5435 \\ \hline 10000065 \end{array}$$

The values of (K) may be found reduced to the seventh position, as follows:—

$$\begin{aligned} \downarrow 1, &= \downarrow 0, 10, \bar{4}, \bar{1}, \bar{9}, \bar{5}, \bar{1}, = \downarrow 0, 0, 0, 0, 0, 0, 953195, (a) \\ \downarrow 0, 1, &= \downarrow 0, 0, 10, 0, \bar{4}, \bar{4}, \bar{7}, = \downarrow 0, 0, 0, 0, 0, 0, 99513, (b) \\ (K), \downarrow 0, 0, 1, &= \downarrow 0, 0, 0, 10, 0, 0, \bar{4}, = \downarrow 0, 0, 0, 0, 0, 0, 9996, (c) \\ \downarrow 0, 0, 0, 1, &= \downarrow 0, 0, 0, 0, 10, 0, 0, = \downarrow 0, 0, 0, 0, 0, 0, 1000, (d) \end{aligned}$$

$$\underline{\downarrow 0,0,0, 1,} = 1000, (d)$$

$$\text{Then, } \downarrow 0,0,0,10, = (d) \times 10 = 10000,$$

$$\text{and, } \downarrow 0,0,0, 0,0,0,\bar{4}, = \underline{-4},$$

$$\therefore \underline{\downarrow 0,0, 1,} = \underline{\downarrow 0,0,0,10,0,0,\bar{4},} = \underline{9996}, (c).$$

$$\text{Then, } \downarrow 0,0,10, = (c) \times 10 = 99960,$$

$$\text{and, } \downarrow 0,0, 0,0,\bar{4},\bar{4},\bar{7}, = \underline{-447},$$

$$\underline{\downarrow 0, 1,} = \underline{\downarrow 0,0,10,0,\bar{4},\bar{4},\bar{7},} = \underline{99513}, (b).$$

$$\text{Then, } \downarrow 0,10, = (b) \times 10 = 995130,$$

$$\text{and, } \downarrow 0, 0,\bar{4}, = (c) \times -4 = -39984,$$

$$\downarrow 0, 0,0,\bar{1},\bar{9},\bar{5},\bar{1}, = \underline{-1951},$$

$$\downarrow 1, = \underline{\downarrow 0,10,\bar{4},\bar{1},\bar{9},\bar{5},\bar{1},} = \underline{953195}, (a).$$

The final values of $\downarrow 1$, $\downarrow 0,1$, $\downarrow 0,0,1$, &c. are also readily found from their positive equalities.

$$\downarrow 1, = \downarrow 0,9,5,7,5,9,8, = \downarrow 0,0,0,0,0,0,953195, (a).$$

$$(K), \downarrow 0,1, = \downarrow 0,0,9,9,5,4,9, = \downarrow 0,0,0,0,0,0, 99513, (b).$$

$$\downarrow 0,0,1, = \downarrow 0,0,0,9,9,9,6, = \downarrow 0,0,0,0,0,0, 9996, (c).$$

$$\underline{\downarrow 0,0,1,} = 9996, (c).$$

$$\text{Then, } \downarrow 0,0,9, = c \times 9 = 89964,$$

$$\text{and, } \downarrow 0,0,0,9,5,4,9, = \underline{9549},$$

$$\therefore \underline{\downarrow 0,1,} = \underline{\downarrow 0,0,9,9,5,4,9,} = \underline{99513}, (b).$$

$$\text{Then, } \downarrow 0,9, = (b) \times 9 = 895617,$$

$$\text{and, } \downarrow 0,0,5, = (c) \times 5 = 49980,$$

$$\text{and, } \downarrow 0,0,0,7,5,9,8, = \underline{7598},$$

$$\therefore \downarrow 1, = \downarrow 0,9,5,7,5,9,8, = 953195, (a).$$

By a similar process, the values of $\downarrow 1$, $\downarrow 0,1$, $\downarrow 0,0,1$, &c. are readily reduced to the eight, or any other position.

$$\downarrow 1, = \downarrow 0,9,5,7,5,9,7,4, = \downarrow 0,0,0,0,0,0,0,9531497, \quad (A).$$

$$\downarrow 0,1, = \downarrow 0,0,9,9,5,4,8,8, = \downarrow 0,0,0,0,0,0,0, \quad 995083, \quad (B).$$

$$\downarrow 0,0,1, = \downarrow 0,0,0,9,9,9,5,5, = \downarrow 0,0,0,0,0,0,0, \quad 99955, \quad (C).$$

$$\downarrow 0,0,1, = \overline{99955}, \quad (C).$$

$$\text{Then, } \downarrow 0,0,9, = (C) \times 9 = 899595$$

$$\text{and } \downarrow 0,0,0,9,5,4,8,8, = \underline{95488}$$

$$\therefore \downarrow 0,1, = \downarrow 0,0,9,9,5,4,8,8, = \underline{995083}, \quad (B).$$

$$\text{Then, } \downarrow 0,9, = (B) \times 9 = 8955747,$$

$$\text{and } \downarrow 0,0,5, = (C) \times 5 = 499775,$$

$$\text{and } \downarrow 0,0,0,7,5,9,7,4, = \underline{75974},$$

$$\therefore \downarrow 1, = \downarrow 0,9,5,7,5,9,7,4, = \underline{9531496}, \quad (A).$$

It will be found convenient to register from 1 to 9 times the values of (A), (B), and (C).

	A.	B.	C.
1	9531497	995083	99955
2	19062994	1990166	199910
3	28594491	2985249	299865
4	38125988	3980332	399820
5	47657485	4975415	499775
6	57188982	5970498	599730
7	66720479	6965581	699685
8	76251976	7960664	799640
9	85783473	8955747	899595

While operating on magnitudes, developed as far as the eight position, the succeeding collection of simple equalities will also be found useful.

$$\begin{aligned}
 2 &= \downarrow 7,2,6,0,7,8,2,6, = 69318201, \text{ in the eight position.} \\
 3 &= \downarrow 11,5,0,4,4,8,6,8, = 109866750, \quad ,, \quad ,, \\
 4 &= \downarrow 14,5,2,2,0,1,1,9, = 138636402, \quad ,, \quad ,, \\
 5 &= \downarrow 16,8,4,8,7,4,4,3, = 160951879, \quad ,, \quad ,, \\
 6 &= \downarrow 18,7,6,5,2,6,9,4, = 179184951, \quad ,, \quad ,, \\
 7 &= \downarrow 20,3,9,8,6,0,1,0, = 194600794, \quad ,, \quad ,, \\
 8 &= \downarrow 21,7,8,2,7,9,4,6, = 207954604, \quad ,, \quad ,, \\
 9 &= \downarrow 23,0,5,0,9,2,9,4, = 219733500, \quad ,, \quad ,, \\
 10 &= \downarrow 24,1,5,1,9,2,9,5, = 230270081, \quad ,, \quad ,,
 \end{aligned}$$

It may be observed, that the whole number 69318201, to which 2 is reduced, is half the number 138636402, to which 4 is reduced, and one third the number 207954604, to which 8 is reduced, because $2^2 = 4$, and $2^3 = 8$. Again, the number 109866750, to which 3 is reduced, is half 219733500, the number to which 9 is reduced, because $3^2 = 9$. Further, since $2 \times 3 = 6$, the numbers representing 2 and 3 added together = the number to which 6 is reduced; and because $2 \times 5 = 10$, the numbers representing 2 and 5 added together = the number to which 10 is reduced, and so on with other numbers. Hence, the only numbers to be calculated, before forming the last collection, are the representatives of 2, 3, 5, and 7.

From what has been previously explained, it is easily found that

$$2 = \downarrow 7,2,6,0,7,8,2,6,$$

$$\text{Then, } 7 \text{ (A)} = 66720479$$

$$2 \text{ (B)} = 1990166$$

$$6 \text{ (C)} = 599730$$

$$\text{To which add } 0,7,8,2,6, = 7826$$

$$\therefore 2 = \underline{\underline{69318201}},$$

Again, $\frac{3}{2} = 1.50000000 = \downarrow 4, 2, 4, 3, 2, 5, 7, 4,$

Then, $4 \text{ (A)} = 38125988,$
 $2 \text{ (B)} = 1990166,$
 $4 \text{ (C)} = 399820,$
 To which add $3, 2, 5, 7, 4, = 32574,$
 $\frac{3}{2} = 40548548,$
 $2 = 69318201,$
 $\therefore 3 = 109866749 = \downarrow 11, 5, 0, 4, 4, 6, 7,$
 $95314970 \sim 10 \text{ (A)},$
 14551779
 $9531497 \sim 1 \text{ (A)},$
 5020282
 $4975415 \sim 5 \text{ (B)},$
 $0, 4, 4, 8, 6, 7,$

$\frac{5}{4} = 1.25000000 = \downarrow 2, 3, 2, 6, 7, 3, 2, 4,$

Then, $2 \text{ (A)} = 19062994,$
 $3 \text{ (B)} = 2985249,$
 $2 \text{ (C)} = 199910,$
 and $6, 7, 3, 2, 4, = 67324,$
 $\frac{5}{4} = 22315477$
 $4 = 138636402$
 $\therefore 5 = 160951879 = \downarrow 16, 8, 4, 8, 7, 4, 4, 3,$
 95314970
 65636909
 57188982
 8447927
 7960664
 487263
 399820
 $8, 7, 4, 4, 3,$

$$\frac{7}{6} = 1.16666667 = \downarrow 1,5,9,0,9,3,3,4,$$

$$\text{Then } 1 \text{ (A)} = 9531497,$$

$$5 \text{ (B)} = 4975415,$$

$$9 \text{ (C)} = 899595,$$

$$\text{and } 09335,$$

$$\text{add } 6. = 179184951,$$

$$194600793, = \downarrow 20,3,9,8,6,0,0,9,$$

$$190629940 \sim 20 \text{ (A)},$$

$$3970853$$

$$2985249 \sim 3 \text{ (B)},$$

$$985604$$

$$899595 \sim 9 \text{ (C)},$$

$$8,6,0,0,9, \text{ and.}$$

Examples in Reduction.

1. *Express 88888.8888 in the eight position,*

$$88888.8888 = 10^4 \times 8 \times 1.11111111$$

$$= 10^4 \times 8 \downarrow 1,1,0,1,0,0,0,1,$$

$$10 = 230270081$$

4

$$10^4 = 921080324$$

$$8 = 207954604$$

$$9531497 \sim 1 \text{ (A)},$$

$$995083 \sim 1 \text{ (B)},$$

$$10001 \text{ and.}$$

$$\therefore 88888.8888 = \downarrow 1139571509,$$

This notation will be explained hereafter.

2. Reduce $\cdot 0066666666$ to the eight position.

$$\begin{aligned}\cdot 0066666666 &= \frac{1}{10^3} \times 6 \times 1'1111111, \\ &= \frac{1}{10^3} \times 6 \downarrow 1,1,0,1,0,0,0,0,\end{aligned}$$

$$\begin{array}{r} 6 = 179184951 \\ \quad 9531497 \text{ I (A)} \\ \quad 995083 \text{ I (B)} \\ \quad 10000 \\ \hline 10 = 230270081 \\ \quad \quad \quad 3 \\ \hline 10^3 = 690810243 \end{array}$$

$$\frac{1}{10^3} = \begin{array}{r} + 189721531 \\ - 690810243 \\ \hline \end{array}$$

$$\therefore \cdot 0066666666 = \downarrow - 501088712,$$

3. What number answers to 1139571509, written $\downarrow 1139571509,$ standing in the eight position?

Four times $230270081 = 921080324$, is a multiple of the value of 10, nearest the given number, but not exceeding the given number.

$$\begin{array}{r} 10^4 = 1139571509 \\ \quad 921080324 \\ \hline \quad 218491185 \\ 8 = 207954604 \\ \hline \quad 10536581 \\ \text{I (A)} = 9531497 \\ \hline \quad 1005084 \\ \text{I (B)} = 995083 \\ \hline \quad 10001 \end{array}$$

$$\therefore 10^4 \times 8 \times \downarrow 1,1,0,1,0,0,0,1, = 88888 \cdot 8888.$$

4. What is the corresponding number of $- 501088712$, written $\downarrow - 501088712,$?

Three times $230270081 = 690810243$ is the nearest multiple of the value of 10 exceeding the given number.

$$10^3 = + 690810243, \\ - 501088712, \text{ given number.}$$

$$\begin{array}{r} 189721531 \\ 6 = 179184951 \end{array}$$

$$\begin{array}{r} 10536580 \\ 1 (A) = 9531497 \end{array}$$

$$\begin{array}{r} 1005083 \\ 1 (B) = 995083 \end{array}$$

$$0,1,0,0,0,0,$$

$$\therefore \frac{1}{10^3} \times 6 \downarrow 1,1,0,1,0,0,0,0, = .0066666666.$$

5. What is the $\frac{17}{43}$ root of π , to nine places of figures?

$\pi = 3 \downarrow 0,4,6,3,1,9,2,9,5, = 114478742$ reduced to the eight position,

$$\begin{array}{r} 3 = 109866750 \\ 4 (B) = 3980332 \\ 6 (C) = 599730 \\ \text{And} \quad 31930 \end{array}$$

$$\begin{array}{r} 114478742 = \\ \text{Multiply by} \quad 17 \end{array}$$

$$43 \mid 1946138614$$

$$45259038 = \downarrow 4,7,16,7,5,1,4, = 1.57234422.$$

PART IV.

OF ANGULAR MAGNITUDES AND
TRIGONOMETRICAL LINES.

To seven places of decimals, $\pi = 3.1415927$; π is generally put for the length of an arc of a circle of 180° , radius = 1.

$$\begin{array}{r}
 312 \overline{)000} \overline{)00} . \\
 \quad 1 \overline{)872} \overline{)00} . \quad \downarrow 0,0,6,9, \\
 \quad \quad 4 \overline{)68} . \\
 \quad \quad \quad 1 . \\
 \hline
 313 \ 876 \ 69 \\
 \quad 282 \ 49 \\
 \quad \quad 11 \\
 \hline
 314 \ 159 \ 29
 \end{array}$$

When the radius is not equal to 1, its length will be mentioned; when the length of the radius is not specified, it is assumed = 1.

Arc of 180° may be represented by $3.12 \downarrow 0,0,6,9,$

„	45°	„	„	$.78 \downarrow 0,0,6,9,$
„	60°	„	„	$1.04 \downarrow 0,0,6,9,$
„	30°	„	„	$.52 \downarrow 0,0,6,9,$
„	15°	„	„	$.26 \downarrow 0,0,6,9,$
„	$7\frac{1}{2}^\circ$	„	„	$.13 \downarrow 0,0,6,9,$
„	$22\frac{1}{2}^\circ$	„	„	$.39 \downarrow 0,0,6,9,$
„	90°	„	„	$1.56 \downarrow 0,0,6,9,$
„	$37\frac{1}{2}^\circ$	„	„	$.65 \downarrow 0,0,6,9,$

The succeeding combinations are also readily established.

Length of an arc of $30^\circ = \cdot 52 \downarrow 0,0,6,9,$

„ „ $3^\circ = \cdot 052 \downarrow 0,0,6,9,$

„ „ $6^\circ = \cdot 104 \downarrow 0,0,6,9,$

„ „ $9^\circ = \cdot 156 \downarrow 0,0,6,9,$

„ „ $12^\circ = \cdot 208 \downarrow 0,0,6,9,$

„ „ $15^\circ = \cdot 26 \downarrow 0,0,6,9,$

„ „ $18^\circ = \cdot 312 \downarrow 0,0,6,9,$

„ „ $21^\circ = \cdot 364 \downarrow 0,0,6,9,$

&c. = &c.

The length of an arc $(a)^\circ$ may be found by saying

$$\text{As } 180^\circ : \pi :: (a)^\circ : \frac{\pi(a)^\circ}{1800}.$$

Other methods to find the length of an arc of a circle corresponding to any number of degrees, minutes, &c. will be given hereafter.

It is well known that if x be the length of an arc of a circle to radius 1, then

$$\sin x = x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \dots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \dots$$

Examples.

1. Find the length of the sine and cosine of an arc of 15° , to seven places of decimals.

$$\text{Length of } 15^\circ = \cdot 26 \downarrow 0,0,6,9,$$

$$(\cdot 26)^2 = \cdot 0676000$$

$$(\cdot 26)^3 = \cdot 0175760$$

$$(\cdot 26)^4 = \cdot 0045698$$

$$(\cdot 26)^5 = \cdot 0011881$$

$$(\cdot 26)^6 = \cdot 0003089$$

2	3	4	5	6	↓ 0,0,36,54
3089	1546	515	129	26	4

2	3	4	↓ 0,0,24,36,
45698	22849	761 6	190 4 ..
		46.. ~ 6	4 6 ..
		7800	195 0
		7.... ~ 9	7
			195 7

$$\begin{array}{r} 2 \\ 676|000 \\ 4056| \sim 6 \end{array}$$

$$\begin{array}{r} \downarrow 0,0,12,18, \\ 338|000 \\ 4|056 \text{ which call A} \\ 22 = 11 \times A \div 2 \end{array}$$

$$\begin{array}{r} 6841|56 \\ 616..| \sim 9 \end{array}$$

$$\begin{array}{r} 3426|78 .. \\ 6|16 .. \text{ call A} \\ 1 .. = 17 \times A \div 2 \\ \hline 342695 \end{array}$$

$$\begin{array}{r} 1\cdot0000000 + \\ 1957 + \\ 342695 - \\ 4 - \\ \hline \cdot 9659258 = \text{cosine of } 15^\circ. \end{array}$$

In finding the cosine, the length of the arc is not required except under the form

$$\cdot 26 \downarrow 0,0,6,9,$$

$$\begin{array}{r}
 \begin{array}{c} 2 \\ 11881 \end{array} \quad \begin{array}{c} 3 \\ 5940 \end{array} \quad \begin{array}{c} 4 \\ 1980 \end{array} \quad \begin{array}{c} 5 \\ 495 \end{array} \Big| \begin{array}{c} 6 \\ 3 \dots \end{array} \quad \begin{array}{c} \downarrow 0,0,30,45, \\ 99 \dots \\ 3 \dots \end{array} \\
 \hline
 102
 \end{array}$$

$$\begin{array}{r}
 \begin{array}{c} 2 \\ 175760 \end{array} \quad \begin{array}{c} 3 \\ 878 \end{array} \Big| \begin{array}{c} 80 \\ 527 \dots \end{array} \quad \begin{array}{c} \downarrow 0,0,18,27, \\ 292 \dots \\ 5 \dots \end{array} \quad \begin{array}{c} 93 \dots \\ 27 \dots = A \\ 4 \dots = 17 \times A \div 2 \end{array} \\
 \hline
 \begin{array}{c} 8947 \dots \\ 80 \dots \end{array} \Big| \begin{array}{c} 2 \\ \dots \end{array} \quad \begin{array}{c} 298 \dots \\ 8 \dots \end{array} \quad \begin{array}{c} 29904 \end{array}
 \end{array}$$

$$\begin{aligned}
 \text{Length of arc} &= \cdot 2617994 + \\
 &\quad 102 + \\
 &\quad \cdot 2618096 \\
 &\quad 29904 - \\
 &\quad \cdot 2588192, \text{ sine of } 15^\circ.
 \end{aligned}$$

RADIUS = 1.

	Length of an arc of 1°, 2°, 3°, 4°, &c.	Length of an Arc of 1', 2', 3', 4', &c.	Length of an Arc of 1'', 2'', 3'', &c.
1	·017453293	·000290888	·000004848
2	·034906585	·000581776	·000009696
3	·052359878	·000872664	·000014544
4	·069813170	·001163552	·000019393
5	·087266463	·001454440	·000024241
6	·104719755	·001745329	·000029089
7	·122173048	·002036216	·000033937
8	·139626340	·002327105	·000038785
9	·157079633	·002617993	·000043633

Examples.

2. Required the length of an arc of a circle to radius 1, of $8^{\circ} 26' 13''$ and its cosine.

$\cdot 13962634$
 $\cdot 00581776$
 $\cdot 00174533$
 $\cdot 00004848$
 $\cdot 00001454$

True to seven places of decimals, $\cdot 1472525 = \text{arc of } 8^{\circ} 26' 13''$.

$$1472525 = 147 \downarrow 0,0,1,7,1,7,$$

$$(\cdot 147)^2 = \cdot 0216090$$

$$(\cdot 147)^4 = \cdot 0004669$$

$\begin{array}{r} 2 \\ 4669 \end{array}$	$\begin{array}{r} 3 \\ 2335 \end{array}$	$\begin{array}{r} 4 \\ 778 \\ \hline 1 \dots \end{array} \sim 1$	$\begin{array}{r} \downarrow 0,0,4,28,4,28, \\ 195 \mid \dots \\ \hline 1 \mid \dots \end{array}$
		$\begin{array}{r} 784 \cdot \mid \\ \hline 6 \dots \end{array} \sim 7$	$\begin{array}{r} 196 \cdot \mid \dots \\ \hline 1 \cdot \mid \dots \end{array}$
			$\hline 197$

$\begin{array}{r} 2 \\ 216 \mid 090 \\ \hline 216 \end{array} \sim 1$	$\begin{array}{r} \downarrow 0,0,2,14,2,14, \\ 108 \mid 045 \\ \hline 216 \end{array}$
$\begin{array}{r} 2165 \mid 22 \\ \hline 152 \dots \end{array} \sim 7$	$\begin{array}{r} 1082 \mid 61 \dots \\ \hline 152 \dots \end{array}$
	$\begin{array}{r} \mid 2 \dots \dots \dots \\ \mid 2 \dots \dots \dots \end{array}$
	$\hline 108417$

$$\begin{array}{r}
 1.0000000 \\
 108417 - \\
 \hline
 \cdot 9891583 \\
 197 + \\
 \hline
 \cdot 9891780
 \end{array}$$

$\cdot 9891780 = \text{cosine } 8^{\circ} 26' 13''$, exact to the last figure.

*Examples of Transformations of the Equations
of condition.*

1. It is required to put $\downarrow 3, \overline{17}, 8, \overline{28}, \overline{12}, 6, \overline{41}$, in a positive convenient form to extract the seventh root of it.

$$\begin{aligned}\downarrow 1, &= \downarrow 0, 10, \overline{4}, \overline{1}, \overline{9}, \overline{5}, \overline{1}, = \downarrow 0, 10, \overline{4}, 0, 0, 0, \overline{1951}, \\ \downarrow 0, 1, &= \downarrow 0, 0, 10, 0, \overline{4}, \overline{4}, \overline{7}, = \downarrow 0, 0, 10, 0, 0, 0, \overline{447}, \\ \downarrow 0, 0, 1, &= \downarrow 0, 0, 0, 10, 0, 0, \overline{4}, = \downarrow 0, 0, 0, 10, 0, 0, \overline{4}, \\ \downarrow 0, 0, 0, 1, &= \downarrow 0, 0, 0, 0, 10, 0, 0, = \downarrow 0, 0, 0, 0, 10, 0, 0,\end{aligned}$$

$$\begin{aligned}&\downarrow 0, 10, \overline{4}, \downarrow 0, 0, 0, \overline{1951}, + 4470 \\ &= \downarrow 0, 0, 96, 0, 0, 0, \overline{6421}, + 384 \\ &= \downarrow 0, 0, 0, 960, 0, 0, \overline{6805},\end{aligned}$$

$$\begin{array}{r} 960000 \\ 6805 \\ \hline 953195 \end{array}$$

$$\begin{aligned}\downarrow 0, 0, 10, 0, 0, 0, \overline{447}, \\ \downarrow 0, 0, 0, 100, 0, 0, \overline{487},\end{aligned}$$

$$\begin{array}{r} 100000 \\ 487 \\ \hline 99513 \end{array}$$

In this manner the following equalities may also be established :

$$\begin{aligned}\therefore \downarrow 1, &= \downarrow 0, 0, 0, 0, 0, 0, 953195, 7 \text{ times} = 6672365 \\ \downarrow 0, 1, &= \downarrow 0, 0, 0, 0, 0, 0, 99513, 7 \text{ times} = 696591 \\ \downarrow 0, 0, 1, &= \downarrow 0, 0, 0, 0, 0, 0, 9996, 7 \text{ times} = 69972 \\ \downarrow 0, 0, 0, 1, &= \downarrow 0, 0, 0, 0, 0, 0, 1000, 7 \text{ times} = 7000\end{aligned}$$

$$\begin{aligned}
 \text{Given, } & \downarrow 3, \overline{17}, 8, \overline{27}, \overline{12}, 6, \overline{41}, \\
 & = \downarrow 0, \overline{13}, \overline{4}, \overline{30}, \overline{39}, \overline{9}, \overline{44}, \\
 & = \downarrow 0, 0, \overline{126}, \overline{30}, \overline{91}, \overline{61}, \overline{135}, \\
 & = \downarrow 0, 0, 0, \overline{1230}, \overline{91}, \overline{61}, \overline{639},
 \end{aligned}$$

$$\begin{array}{r}
 12300 \\
 91 \\
 \hline
 122090 \\
 61 \\
 \hline
 1220290 \\
 639 \\
 \hline
 696591 \mid 1219651 \quad (\downarrow 0, \overline{1}, \overline{7}, \overline{4}, \overline{7}, \overline{5}, \overline{1}, \\
 \quad \quad \quad 696591 \\
 \hline
 69972 \mid 523060 \\
 \quad \quad \quad 489804 \\
 \hline
 7000 \mid 33256 \\
 \quad \quad \quad 28000 \\
 \hline
 700 \mid 5256 \\
 \quad \quad \quad 4900 \\
 \hline
 70 \mid 356 \\
 \quad \quad \quad 350 \\
 \hline
 7 \mid 6 \\
 \quad \quad \quad 7 \text{ nearly.}
 \end{array}$$

$\therefore \downarrow 0, \overline{1}, \overline{7}, \overline{4}, \overline{7}, \overline{5}, \overline{1}, =$ the seventh root of $\downarrow 3, \overline{17}, 8, \overline{27}, \overline{12}, 6, \overline{41},$

or, $\downarrow 0, \overline{7}, \overline{49}, \overline{28}, \overline{49}, \overline{35}, \overline{7}, = \downarrow 3, \overline{17}, 8, \overline{27}, \overline{12}, 6, \overline{41},$

$= \downarrow 0, 0, 0, 0, 0, 0, \overline{1219651},$ which may readily be reduced to

$$\downarrow 1, \overline{2}, \overline{6}, \overline{7}, \overline{4}, \overline{5}, \overline{4},$$

$$\begin{array}{r}
 953195 \mid 1219651 \quad (1, \\
 \quad \quad \quad 953195 \\
 \hline
 266456
 \end{array}$$

$$\begin{array}{r} 99513 \) \ 266\ 45\ 6(\ 2, \\ \underline{199\ 02\ 6} \end{array}$$

$$\begin{array}{r} 9996 \) \ 67\ 43\ 0(\ 6, \\ \underline{59\ 97\ 6} \end{array}$$

$$\underline{7,4,5,4,}$$

2. Put $\downarrow 3, \overline{17}, 8, \overline{27}, \overline{12}, 6, \overline{41}$, in a form so that the square root of it is represented in whole numbers.

$$953195 \times 2 = 1906390; 99513 \times 2 = 199026; 9996 \times 2 = 19992.$$

The given expression, when reduced to the seventh place, becomes

$$\downarrow 0,0,0,0,0,0,1219651,$$

$$\begin{array}{r} 199026 \) \ 12196\ 5\ 1(\downarrow 0,6,1, \\ \underline{1194\ 15\ 6} \end{array}$$

$$\begin{array}{r} 19992 \) \ 25\ 49\ 5 \\ \underline{19992} \end{array}$$

$$\begin{array}{r} 2 \) \ 5503 \\ \underline{4} \end{array}$$

$$\underline{2,7,5,1,}$$

$\downarrow 0,6,1,2,7,5,1$, represents the square root of $\downarrow 3, \overline{17}, 8, \overline{27}, \overline{12}, 6, \overline{41}$,

Find the cosine of $6^\circ 26' 23''.5$.

$$10471976 = 6^\circ$$

$$00581776 = 20'$$

$$00174533 = 6''$$

$$00009696 = 20'''$$

$$00001454 = 3''''$$

$$00000242 = .5''''$$

$$\cdot 1123968 \cdot = \text{arc of } 6^\circ 26' 23''.5$$

$$1123968 = 112 \downarrow 0,0,3,5,3,8,$$

$$\cdot 112^2 = \cdot 0125440$$

$$\cdot 112^4 = \cdot 0001574$$

$$\begin{array}{ccccccc} & 2 & & 3 & & 4 & & \downarrow 0,0,12,20,12,32, \\ 1574 & & 787 & & 262 & . & 66 & . \dots \\ & & & & 1 & \dots & 1 & . \dots \\ & & & & & & \underline{67} \end{array}$$

2 divided into 1574, gives 787; 3 divided into 787, gives 262; and 4 divided into 262, gives 65.5 or 66. Then, $\downarrow 0,0,12,20,12,32$, operates on 66; the extent to which the operation is carried only requires the use of one of these factors, namely $\downarrow 0,0,12$.

$$\begin{array}{r} 2 \\ 1254 \overline{) 40} \\ 376 \cdot \end{array} \sim 3$$

$$\begin{array}{r} 12619 \overline{) 4} \\ 63 \dots \end{array} \sim 5$$

$$\begin{array}{r} \downarrow 0,0,6,10,6,16, \\ 627 \overline{) 20} \cdot \\ 376 \cdot \\ 1 \cdot \end{array}$$

$$\begin{array}{r} 6309 \overline{) 7} \cdot \cdot \cdot \\ 63 \overline{) 3} \cdot \cdot \cdot \cdot \cdot \\ 4 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ 1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \end{array}$$

$$63165$$

$$\begin{array}{r} 1.0000000 \\ 63165 \end{array}$$

$$\begin{array}{r} .9936835 \\ 67 \end{array}$$

$$\underline{\underline{.9936902}} = \cos 6^\circ 26' 23'' \cdot 5$$

3. What is the length of an arc of $9^\circ 20' 45''$, and its sine and cosine, radius = 1?

$$\begin{array}{r} 1570796 - 9^\circ \\ 58178 - 20' \\ 1939 - 40'' \\ 242 - 5'' \end{array}$$

$$163 \downarrow 0,0,0,7,0,8, = \underline{\underline{1631155}}$$

The numbers employed to find the square, cube, &c., of .163 may be taken from the following line containing from 1 to 9 times 163. It may be observed that only 15 numbers are required to be taken this line.

1	2	3	4	5	6	7	8	9
163	326	489	652	815	978	1141	1304	1467

$$\begin{array}{r} 489 \\ 978 \\ \hline 163 \end{array}$$

$$\cdot 0265690 = (\cdot 163)^2$$

$$\begin{array}{r} 1467 \\ 978 \\ 815 \\ 978 \\ \hline 326 \end{array}$$

$$\begin{array}{r} \cdot 004330747 \\ \cdot 0043307 = (\cdot 163)^3 \end{array}$$

$$\begin{array}{r} 1141 \\ 4890 \\ 489 \\ 652 \\ \hline \end{array}$$

$$\begin{array}{r} \cdot 0007059041 \\ 0007059 = (\cdot 163)^4 \end{array}$$

$$\begin{array}{r} 1467 \\ 815 \\ \hline 11410 \end{array}$$

$$\begin{array}{r} \cdot 0001150617 \\ \cdot 0001151 = (\cdot 163)^5 \end{array}$$

$$\begin{array}{r} 2 \\ 1151 \end{array}$$

$$\begin{array}{r} 3 \\ 576 \end{array}$$

$$\begin{array}{r} 4 \\ 192 \end{array}$$

$$\begin{array}{r} 5 \\ 48 \end{array}$$

$$10$$

$$\begin{array}{r} 2 \\ 43307 \end{array}$$

$$\begin{array}{r} 3 \\ 21654 \\ 15 \dots \sim 7 \end{array}$$

$$\begin{array}{r} \downarrow 0,0,0,21,0,24, \\ 7218 \mid \dots \\ 15 \mid \dots \\ \hline 7233 \end{array}$$

Length of arc = 1631155

7233 -

10 +

$$\hline 1623932 = \sin 9^\circ 20' 45''.$$

$$\begin{array}{r} 2 \\ 7059 \end{array}$$

$$\begin{array}{r} 3 \\ 3530 \end{array}$$

$$\begin{array}{r} 4 \\ 1177 \mid \\ 8 \dots \sim 7 \end{array}$$

$$\begin{array}{r} \downarrow 0,0,0,28,0,32, \\ 294 \mid \dots \\ 1 \mid \dots \\ \hline 295 \end{array}$$

$$\begin{array}{r} 2 \\ 2656 \mid 90 \\ 186 \dots \sim 7 \end{array}$$

$$\begin{array}{r} 266062 \mid \\ 2 \dots \sim 8 \end{array}$$

$$\downarrow 0,0,0,14,0,16,$$

$$\begin{array}{r} 1328 \mid 45 \dots \\ 186 \dots \end{array}$$

$$\begin{array}{r} 133031 \mid \dots \\ 2 \mid \dots \end{array}$$

$$\hline 133033$$

$$\begin{array}{r}
 1.0000000 \\
 133033 - \\
 \hline
 .9866967 \\
 295 + \\
 \hline
 .9867262 = \cos 9^\circ 20' 45''.
 \end{array}$$

4. The length of an arc of $35^\circ 40'' = .6225008$; find the cosine.

$$62 \downarrow 0,0,4,0,2,7,5, = 6225008.$$

	62
	62
	124
	372
Square	3844 ~ 1
	7688 ~ 2
	11532 ~ 3
	15376 ~ 4
	19220 ~ 5
	23064 ~ 6
	26908 ~ 7
	30752 ~ 8
	34596 ~ 9

The following are taken from the above column:—

$$\begin{array}{r}
 1537 \overline{)6} \\
 1537 \overline{)6} \\
 30752 \\
 11532 \\
 \hline
 .1477634 \text{ fourth.}
 \end{array}$$

$$\begin{array}{r}
 1 \overline{)5376} \\
 11 \overline{)532} \\
 230 \overline{)64} \\
 2690 \overline{)8} \\
 26908 \\
 15376 \\
 3844 \\
 \hline
 .0568003 \text{ sixth.}
 \end{array}$$

$$\begin{array}{r}
 1 \overline{)1532} \\
 3075 \overline{)200} \\
 23064 \\
 19220 \\
 \hline
 0218340 \text{ eighth.}
 \end{array}$$

2	3	4	5	6	7	8	
218340	109170	36390	9098	1820	304	43	5

The 5 in the seventh decimal place, here produced by dividing 218340 continually by 1 ~ 8 when operated upon by $\downarrow 0,0,32, \dots$ is not increased a unit, so the operation is omitted.

$$\begin{array}{r}
 \begin{array}{ccccc}
 2 & 3 & 4 & 5 & 6 \\
 568003 & 284002 & 94667 & 23667 & 4733 \\
 & & & & 19 \dots \sim 4
 \end{array}
 \quad \downarrow 0,0,24,0,12, \\
 \begin{array}{r}
 789 \dots \\
 19 \dots \\
 \hline
 808
 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \begin{array}{ccc}
 2 & 3 & 4 \\
 1477634 & 738817 & 2462 \mid 72 \\
 & & 985 \cdot \mid \sim 4
 \end{array}
 \quad \downarrow 0,0,16,0,8,28,20, \\
 \begin{array}{r}
 615 \mid 68 \cdot \\
 9 \mid 85 \cdot \\
 7 \cdot \\
 \hline
 625 \ 60 \mid \dots \dots \cdot \\
 5 \mid \dots \dots \cdot \\
 625 \ 65 \cdot \mid \dots \dots \cdot \\
 2 \cdot \mid \dots \dots \cdot \\
 \hline
 625 \ 67
 \end{array}
 \end{array}$$

$$\begin{array}{r}
 250260 \cdot \mid \\
 2 \dots \dots \cdot \mid \sim 7
 \end{array}$$

$$\begin{array}{r}
 \begin{array}{cc}
 2 & \\
 384400 \mid 0 \dots \\
 15376 \dots \mid \sim 4
 \end{array}
 \quad \downarrow 0,0,8,0,4,14,10, \\
 \begin{array}{r}
 192 \mid 200 \mid 0 \dots \\
 1 \mid 537 \mid 6 \dots \\
 5 \mid 4 \dots \\
 \hline
 193 \ 74 \mid 30 \mid \dots \\
 7 \mid 8 \dots \dots \cdot \\
 2 \mid 5 \dots \dots \cdot \\
 2 \mid \dots \dots \cdot \\
 \hline
 193 \ 75 \ 3 \ 5
 \end{array}
 \end{array}$$

$$1.0000000 +$$

$$1937535 -$$

$$\cdot 8062465$$

$$62567 +$$

$$\cdot 8125032$$

$$808 -$$

$$\cdot 8124224$$

$$5 +$$

$$\cdot 8124229 = \cos 35^\circ 40'.$$

5. *It is readily found that the length of an arc of $35^{\circ} 38' 49''$ = 6221566, find the cosine.*

$$62 \downarrow 0,0,3,4,7,4, = 6221566.$$

Many of the results, and the column of the last example may be employed here, as 62 are the two first figures.

$$\begin{array}{r} 6 \\ 4734 \overline{) 14 \dots} \sim 3 \\ 4818 \overline{) 2 \dots} \sim 4 \end{array}$$

$$\downarrow 0,0,18,24,42,$$

$$\begin{array}{r} 789 \overline{) \dots} \\ 14 \overline{) \dots} \end{array}$$

$$\begin{array}{r} 803 \overline{) \dots} \\ 2 \overline{) \dots} \end{array}$$

$$805 \overline{) \dots}$$

$$\begin{array}{r} 4 \\ 2462 \overline{) 72} \\ 739 \overline{) \dots} \sim 3 \end{array}$$

$$\downarrow 0,0,12,16,28,16,$$

$$\begin{array}{r} 615 \overline{) 68} \\ 7 \overline{) 39} \\ 4 \end{array}$$

$$\begin{array}{r} 24924 \overline{) 4} \\ 100 \overline{) \dots} \sim 4 \end{array}$$

$$\begin{array}{r} 623 \overline{) 11 \dots} \\ 10 \overline{) 0 \dots} \\ 17 \overline{) \dots} \\ 1 \overline{) \dots} \end{array}$$

$$62429$$

$$\downarrow 0,0,6,8,14,8,$$

$$\begin{array}{r} 192 \overline{) 2000 \dots} \\ 1153 \overline{) 2 \dots} \\ 29 \end{array}$$

$$\begin{array}{r} 1933 \overline{) 561} \\ 1547 \overline{) \dots} \\ 1 \end{array}$$

$$\begin{array}{r} 38702 \overline{) 26} \\ 271 \overline{) \dots} \sim 7 \end{array}$$

$$\begin{array}{r} 1935 \overline{) 109 \dots} \\ 27 \overline{) 1 \dots} \\ 15 \overline{) \dots} \end{array}$$

$$1935395$$

$$\begin{array}{r} 1.0000000 + \\ 1935395 - \\ \hline \end{array}$$

$$\begin{array}{r} .8064605 \\ 62429 + \\ \hline \end{array}$$

$$\begin{array}{r} .8127034 \\ 805 - \\ \hline \end{array}$$

$$\begin{array}{r} .8126229 \\ 5 + \\ \hline \end{array}$$

$$.8126234 = \cos 35^\circ 38' 49''.$$

6. What is the cosine $37^\circ 39' 49''.5$, the length of the arc being $= .6573565$?

To calculate the length of an arc of any number of degrees, minutes, the following table is easily constructed, and readily applied;

$.0000048481368$ being the length of an arc of $1''$, to radius unity:

$$.0000048481368 \sim 1$$

$$.0000096962736 \sim 2$$

$$.0000145444104 \sim 3$$

$$.0000193925472 \sim 4$$

$$.0000242406840 \sim 5$$

$$.0000290888208 \sim 6$$

$$.0000339369576 \sim 7$$

$$.0000387850944 \sim 8$$

$$.0000436332312 \sim 9$$

$$37^\circ 39' 49''.5 = 135589''.5.$$

$$.4848137 = 100000''$$

$$.1454441 = 30000$$

$$242407 = 5000$$

$$24241 = 500$$

$$3879 = 80$$

$$436 = 9$$

$$24 = .5$$

$$657 \downarrow 0,0,0,5,4,2,4, = .6573565$$

The numbers whose sums give the square, cube, &c. of $\cdot 657$, may be found by inspection, when the following line is formed.

1	2	3	4	5	6	7	8	9
657	1314	1971	2628	3285	3942	4599	5256	5913

4599	5913
3285	2628
3942	3942
	657
	1971
	2628
<u>4316490</u> square.	<u>2835934</u> cube

2628	5913
1971	1314
5913	40
3285	1971
1971	3942
5256	5256
1314	657
<u>1863209</u> fourth.	<u>1224128</u> fifth.

5256	1314
1314	657
657	2628
2628	1314
1314	1314
1314	657
<u>0804252</u> sixth.	<u>0528394</u> seventh.

2628	
5913	
1971	
5256	
1314	
3285	
<u>0347155</u> eight.	

2 3 4 5 6 7 8
347155 173578 57859 14465 2893 482 69 9.

2 3 4 5 6 ↓ 0,0,0,5,4,2,4,
804252 402126 134042 33510 6702 III7|....
3.... 3|....
II20

2 3 4 ↓ 0,0,0,20,16,8,16,
1863209 931605 31053|5 7763|4...
155...| ~ 5 15|5...
311156| 77789|.....
12.....| ~ 4 12|.....
311204.| 77801.|.....
6.....| 1|.....
77802

2 ↓ 0,0,0,10,8,4,8,
4316|490 2158|245.
2158.| ~ 5 2|158.
I.
21604|04..
17|3.....
9|.....
1|.....
2160587

1'0000000 +
2160587 -

7839412
77802 +

7917214
1120 -

7916094
9 +

7916103 = cos 37° 39' 49".5.

PROBLEM.

7. Given the apparent altitude of the moon's centre $8^{\circ} 26' 13''$ (a), the true altitude, $9^{\circ} 20' 45''$ (A), the apparent altitude of a star $35^{\circ} 40'$ (a_1), the true altitude, $35^{\circ} 38' 49''$ (A_1), and the apparent distance $31^{\circ} 13' 26''$ (d); required the true distance, so as to find the longitude at sea.

Put D = the true distance.

A, A_1 = the true altitudes

d = the apparent distance.

a, a_1 = apparent altitudes.

It is established by writers on Spherical Trigonometry, that,

$$\cos D = [\cos d + \cos (a + a_1)] \frac{\cos A \cos A_1}{\cos a \cos a_1} - \cos (A + A_1).$$

$$\begin{array}{rcl} d \ 31^{\circ} \ 13' \ 26'' & \cos = & \cdot 8551482 \\ (a + a_1) \ 44 \ \ 6 \ 13 & \cos = & \cdot 7180824 \\ & & \hline & & 1\cdot5732306 \end{array}$$

$$\begin{array}{rcl} (A) \ 9^{\circ} \ 20' \ 45'' & \cos = & \cdot 9867261 \\ (a) \ 8 \ 26 \ 13 & \cos = & \cdot 9891779 \end{array}$$

$$\begin{array}{r} 9867261 \\ 19735 \\ \hline 9887006 \end{array} \quad \begin{array}{l} \downarrow 0,0,2,4,8,2,8, \\ \dots \\ \dots \\ \dots \end{array}$$

$$\begin{array}{r} 9887006 \\ 3955 \\ 791 \\ \hline 9891779 \end{array}$$

$$9891779$$

$$(A,) 35^{\circ} 38' 49'' \cos = \cdot 8126236$$

$$(a,) 35 40 0 \cos = \cdot 8124229$$

$$\begin{array}{r} 8124 \overline{229} . \\ 16 \overline{25} \dots \downarrow 0,0,0,2,4,7,0, \\ 3 \overline{25} \dots \dots \\ 5 \overline{7} \dots \dots \end{array}$$

$$\downarrow 0,0,0,2,4,7,0,$$

$$\downarrow 0,0,2,4,8,2,8,$$

$$\downarrow 0,0,2,4,8,2,8,$$

$$8126236$$

$$\therefore \frac{\downarrow 0,0,0,2,4,7,0,}{\downarrow 0,0,2,4,8,2,8,} = 0,0,\overline{2},\overline{2},\overline{4},\overline{5},\overline{8},$$

Then,

$$15732306 \downarrow 0,0,\overline{2},\overline{2},\overline{4},\overline{5},\overline{8}, = 1\cdot 5697186$$

$$\cdot 7071959 \cos (A + A,) 44^{\circ} 59' 34''.$$

$$\cdot 8625227 \cos (D) \quad 31^{\circ} 23' 56''\cdot 3.$$

8. *The apparent distance of the moon's centre from the star Regulus was $63^{\circ} 35' 14''$ (d), the apparent altitude of the moon's centre, $24^{\circ} 29' 44''$ (a), the apparent altitude of the star, $45^{\circ} 9' 12''$ ($a,$), the true altitude of the moon's centre, $25^{\circ} 17' 45''$ (A), and the true altitude of the star, $45^{\circ} 8' 15''$ ($A,$); required the true distance (D), so as to find the longitude at sea.*

$$\cos D = [\cos d + \cos (a + a,)] \frac{\cos A \cos A,}{\cos a \cos a,} - \cos (A + A,).$$

$$(d) 63^{\circ} 35' 14'' \cos = \cdot 4448349$$

$$(a + a,) 69 38 56 \cos = \cdot 3477722$$

$$\cdot 7926071$$

$$(A) 25^{\circ} 17' 45'' \cos = \cdot 9041135$$

$$(a) 24^{\circ} 29' 44'' \cos = \cdot 9099935$$

$$(A,) 45^{\circ} 8' 15'' \cos = \cdot 7054078$$

$$(a,) 45 9 12 \cos = \cdot 7052119$$

$$\therefore \cos A \downarrow 0,0,6,4,8,5,6, = \cos a, \text{ and } \cos a, \downarrow 0,0,0,2,7,7,9, = \cos A,;$$

$$\frac{\downarrow 0,0,0,2,7,7,9,}{\downarrow 0,0,6,4,8,5,6,} = 0,0,\overline{6},\overline{2},\overline{1},\overline{2},\overline{3},$$

$$\begin{array}{r}
 .792\ 607\ 1 \dots\dots\dots \\
 1\ 6 \dots\dots\dots \\
 2 \dots\dots\dots \\
 \hline
 \downarrow 0,0,0,0,0,2,3, = 792\ 608\ 9 \dots + 1 \\
 4\ 7\ 5\ 5\ 6 \dots - 6 = \beta \\
 1\ 6\ 6 \dots + 7 \times \beta \div 2
 \end{array}$$

$$\begin{array}{r}
 \downarrow 0,0,\bar{6},0,0,2,3, = 787\ 8\ 6\ 9\ 9 \dots \\
 \phantom{\downarrow 0,0,\bar{6},0,0,2,3, =} 1\ 5\ 7\ 6 \dots - 2 \\
 \phantom{\downarrow 0,0,\bar{6},0,0,2,3, =} 7\ 8 \dots - 1
 \end{array}$$

$$\downarrow 0,0,\bar{6},\bar{2},\bar{1},2,3, = \underline{787\ 7\ 0\ 4\ 5} = \cos 63^\circ 4' 35'', (D).$$

When half the indices to the right of $\downarrow$ are found, the remainder may be determined by common division.

$$\begin{array}{r}
 \text{From } 7054078 \\
 \text{take } 7052119 \\
 \hline
 705\ 2 \quad 1959 \\
 1410 (2 \\
 \hline
 70\ 5 \quad 549 (7 \\
 494 \\
 \hline
 7\ 1 \quad 56 (7 \\
 50 \\
 \hline
 6 (9 \\
 6 \\
 \hline

 \end{array}$$

9. Suppose the apparent distance between the centres of the sun and moon to be $83^\circ 57' 33''$ (d), the apparent altitude of the moon's centre, $27^\circ 34' 5''$ (a), the apparent altitude of the sun's centre, $48^\circ 27' 32''$ (a_1), the true altitude of the moon's centre, $28^\circ 20' 48''$ (A), and the true altitude of the sun's centre, $48^\circ 26' 49''$ (A_1); find the true distance (D), so as to determine the longitude at sea.

$$\cos D + [\cos d + \cos (a + a_1)] \frac{\cos A \cos A_1}{\cos a \cos a_1} - \cos (A + A_1).$$

$$\begin{array}{r}
 (d) \ 83^\circ 57' 33'' \cos = .1052372 \\
 (a + a_1) \ 76 \quad 1 \quad 37 \cos = .2414656 \\
 \hline
 .3467028
 \end{array}$$

$$(A) 28^{\circ} 20' 48'' \cos = .8800909$$

$$(a) 27 \ 34 \ 5 \cos = .8864618$$

$$\begin{array}{r} 880 \overline{) 0909} \dots \downarrow 0,0,7,2,1,6,3, \\ 6 \overline{) 1606} \dots \\ 1 \overline{) 85} \dots \end{array}$$

$$\begin{array}{r} 8862 \overline{) 700} \dots \\ 1 \overline{) 773} \dots \\ 8 \overline{) 9} \dots \\ 5 \overline{) 3} \dots \\ 3 \overline{) \dots} \end{array}$$

$$\begin{array}{r} 8862 \overline{) 700} \dots \\ 1 \overline{) 773} \dots \\ 8 \overline{) 9} \dots \\ 5 \overline{) 3} \dots \\ 3 \overline{) \dots} \end{array}$$

$$\begin{array}{r} 8862 \overline{) 700} \dots \\ 1 \overline{) 773} \dots \\ 8 \overline{) 9} \dots \\ 5 \overline{) 3} \dots \\ 3 \overline{) \dots} \end{array}$$

$$\begin{array}{r} 8862 \overline{) 700} \dots \\ 1 \overline{) 773} \dots \\ 8 \overline{) 9} \dots \\ 5 \overline{) 3} \dots \\ 3 \overline{) \dots} \end{array}$$

$$\begin{array}{r} 8862 \overline{) 700} \dots \\ 1 \overline{) 773} \dots \\ 8 \overline{) 9} \dots \\ 5 \overline{) 3} \dots \\ 3 \overline{) \dots} \end{array}$$

$$\begin{array}{r} 8862 \overline{) 700} \dots \\ 1 \overline{) 773} \dots \\ 8 \overline{) 9} \dots \\ 5 \overline{) 3} \dots \\ 3 \overline{) \dots} \end{array}$$

$$\begin{array}{r} 8862 \overline{) 700} \dots \\ 1 \overline{) 773} \dots \\ 8 \overline{) 9} \dots \\ 5 \overline{) 3} \dots \\ 3 \overline{) \dots} \end{array}$$

$$\begin{array}{r} 8862 \overline{) 700} \dots \\ 1 \overline{) 773} \dots \\ 8 \overline{) 9} \dots \\ 5 \overline{) 3} \dots \\ 3 \overline{) \dots} \end{array}$$

$$(A) 48^{\circ} 26' 49'' \cos = .6633133 \quad \downarrow 0,0,0,2,3,5,3,$$

$$(a) 48 \ 27 \ 32 \cos = .6631573$$

$$\begin{array}{r} 6631 \overline{) 573} \dots \\ 1 \overline{) 326} \dots \\ 1 \overline{) 99} \dots \\ 3 \overline{) 3} \dots \\ 2 \overline{) \dots} \end{array}$$

$$\begin{array}{r} 6631 \overline{) 573} \dots \\ 1 \overline{) 326} \dots \\ 1 \overline{) 99} \dots \\ 3 \overline{) 3} \dots \\ 2 \overline{) \dots} \end{array}$$

$$\begin{array}{r} 6631 \overline{) 573} \dots \\ 1 \overline{) 326} \dots \\ 1 \overline{) 99} \dots \\ 3 \overline{) 3} \dots \\ 2 \overline{) \dots} \end{array}$$

$$\begin{array}{r} 6631 \overline{) 573} \dots \\ 1 \overline{) 326} \dots \\ 1 \overline{) 99} \dots \\ 3 \overline{) 3} \dots \\ 2 \overline{) \dots} \end{array}$$

$$\begin{array}{r} 6631 \overline{) 573} \dots \\ 1 \overline{) 326} \dots \\ 1 \overline{) 99} \dots \\ 3 \overline{) 3} \dots \\ 2 \overline{) \dots} \end{array}$$

$$\frac{\downarrow 0,0,0,2,3,5,3,}{\downarrow 0,0,7,2,1,6,3,} = \downarrow 0,0,7,0,2,1,0,$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$\begin{array}{r} 346 \overline{) 7028} \dots + \\ 2 \overline{) 4269} \dots - \\ 9 \overline{) 7} \dots + \end{array}$$

$$(A + A) 76^{\circ} 47' 37'' \cos = \frac{3442922}{2284592}$$

$$\text{True distance, (D) } 83 \ 20 \ 53.98 \cos = \frac{1158330}{1158330}$$

This is one of the model examples given by writers on the Longitude.

10. *The natural cosine of an angle is given equal .1158330; how many degrees, minutes, and seconds are contained in the arc?*

It is well known, if c be the length of a circular arc to radius 1, and s its sine, then

$$c = s + \frac{1}{2} \frac{s^3}{3} + \frac{1}{2} \frac{3}{4} \frac{s^5}{5} + \frac{1}{2} \frac{3}{4} \frac{5}{6} \frac{s^7}{7} + \&c.$$

$$1158330 = 115 \downarrow 0,0,7,2,2,$$

The square of .115 = .013225; once, twice, three times, &c. of 13225, or of any other number, may be found without much mental exertion, as follows:

1	~	13225	
2	~	26450	
			~ 39675 ~ 3
4	~	52900	79350 ~ 6
8	~	105800	119025 ~ 9
		13225	52900
7	~	92575	66125 ~ 5

Twice one = 2, twice 2 = 4, twice 4 = 8; take 1 from 8, and 7 remains; 1 and 2 make 3, twice 3 = 6; 6 plus 3 = 9; and finally, 9 minus 4 = 5; or, 5 times 13225 = half 132250 = 66125.

$$.013225 \times .115 = (.115)^3$$

$$\begin{array}{r} 661 \overline{) 25} \\ 1322 \overline{) 5} \\ 13225 \end{array}$$

.0015209... = (.115)³, true to seven places of decimals.

The numbers added together are taken from the multiples of 13225, so easily found above.

$$\cdot 0015209 \times (\cdot 013225) = (\cdot 115)^5$$

$$\begin{array}{r} 119025 \\ 2 \overline{) 64500} \\ 66 \overline{) 125} \\ 132 \overline{) 25} \end{array}$$

$$\cdot 0000201 \dots = (\cdot 115)^2$$

$\frac{1}{2}$ 15209	3 $\begin{array}{r} 760 \overline{) 5} \\ 53 \dots \end{array} \sim 7$	$\downarrow 0,0,21,6,6,$ $\begin{array}{r} 253 \overline{) 5} \dots \\ 5 \overline{) 3} \dots \\ \hline 2588 \overline{) \dots} \\ 2 \overline{) \dots} \\ \hline 2590 \end{array}$
------------------------	---	--

Operating with $\downarrow 0,0,0,0,6$, would not increase the result 2590 by a unit, so it was only necessary to employ $\downarrow 0,0,21,6$,

$\frac{1}{2}$ 201	$\frac{3}{4}$ 101	5 $\begin{array}{r} 75 \overline{) \dots} \\ 5 \dots \end{array} \sim 7$	$\downarrow 0,0,35,10,10,$ $\begin{array}{r} 15 \overline{) \dots} \\ 1 \overline{) \dots} \\ \hline 16 \end{array}$
----------------------	----------------------	---	---

$$\begin{array}{r} 1158330 \\ 2590 \\ 16 \\ \hline \end{array}$$

$$\cdot 1160936 \text{ arc of } 6^\circ 39' 6''$$

$$\begin{array}{r} \text{From } 90^\circ \quad 0' \quad 0'' \\ \text{take } \quad 6 \quad 39 \quad 6 \\ \hline \end{array}$$

$$83 \quad 20 \quad 54, \text{ true distance.}$$

The degrees, &c. corresponding to the length of an arc, may be found by any of the given rules or by the small tables, pages 62, 72, or by direct calculation, thus:—

As $3.1415927 : 180^\circ :: .1160936 : 6^\circ 39' 6''$.

Because $3.12 \downarrow 0,0,6,9, = 3.1415927$, nearly.

$\therefore \frac{180}{3.12} \times .1160936 \downarrow 0,0,\bar{6},\bar{9}, =$ degrees, and decimal parts of a degree, in the arc. This expression may be reduced to

$$\begin{aligned} & \frac{18000}{312} \times .1160936 \downarrow 0,0,\bar{6},\bar{9}, \\ & = \frac{750}{13} \times .1160036 \downarrow 0,0,\bar{6},\bar{9}, \end{aligned}$$

In general terms, if L be the length of an arc to radius 1, the degrees, and decimal parts of a degree, in this arc will be expressed by

$$\frac{750}{13} \times L \downarrow 0,0,\bar{6},\bar{9},$$

$$\begin{array}{r} 4) 116.0936 \quad 1000 \text{ times} \\ \quad 29.0234 \quad 250 \text{ times} \\ \hline 13) 87.0702 \quad 750 \text{ times} \\ \quad 6.69|77. \quad + \\ \quad \quad 4|02. \quad - 6 = A \\ \quad \quad \quad 1. \quad + 7 \times A \div 2 \\ \hline \quad 6.657|6 \dots + \\ \quad \quad 59 \dots - 9 \\ \hline \quad 6.6517 \\ \quad \quad 60 \\ \hline \quad 39.1020 \\ \quad \quad 60 \\ \hline \end{array}$$

$6^\circ 39' 6'' \quad 6.1200$, as before found.

This method will be found useful when tables are not convenient.

11. *The length of a circular arc is .165433; how many degrees, &c. does it contain?*

$$\begin{array}{r}
 4 \overline{) 165.433} \\
 \underline{41.358} \\
 13 \overline{) 124.075} \\
 \underline{9.544} \quad \dots \\
 \quad 57 \quad \dots \sim 6 \\
 \quad \quad 8 \quad \dots \sim 9 \\
 \hline
 9^{\circ} 47' 9'' \\
 \quad 60 \\
 \hline
 28' 7'' 40''' \\
 \quad 60 \\
 \hline
 44'' 400'''
 \end{array}$$

$9^{\circ} 28' 44''$ nearly in an arc whose length is .165433. The most lengthy and accurate calculation makes the degrees in this arc $9^{\circ} 28' 43''$. It is easily observed, that the simple method here introduced gives the degrees, &c. in an arc with considerable accuracy, even when only three places of decimals are employed, as in the last example.

12. *The sine of an arc is given = .342025; find the degrees, &c. contained in the arc.*

$$342 \downarrow 0,0,0,0,6, = .342025.$$

By the help of the following line, which may be computed in the usual manner before explained, the cube, fifth, seventh, and ninth powers of .342 are readily determined.

1	2	3	4	5	6	7	8	9
342	684	1026	1368	1710	2052	2394	2736	3078

.0400017 cube.

.0046788 fifth.

.0005472 seventh.

.0000640 ninth.

$$\begin{array}{cccccc} \frac{1}{2} & \frac{3}{4} & \frac{5}{6} & \frac{7}{8} & 9 & \downarrow 0,0,0,0,54, \\ 640 & 320 & 240 & 200 & 175 & 19 \dots | \dots \dots \end{array}$$

$$\begin{array}{cccccc} \frac{1}{2} & \frac{3}{4} & \frac{5}{6} & 7 & & \downarrow 0,0,0,0,42, \\ 5472 & 2736 & 2052 & 1710 & & 244 \dots | \dots \dots \end{array}$$

$$\begin{array}{cccccc} \frac{1}{2} & & \frac{3}{4} & 5 & & \downarrow 0,0,0,0,30, \\ 46788 & & 23394 & 17546 & & 3509 \dots | \dots \dots \\ & & & & & 1 \dots | \dots \dots \\ & & & & & \hline & & & & & 3510 \end{array}$$

$$\begin{array}{cccccc} \frac{1}{2} & & 3 & & & \downarrow 0,0,0,0,18, \\ 400017 & & 200009 & & & 66670 \dots | \dots \dots \\ & & 12 \dots \dots | \sim 6 & & & 12 \dots \dots \\ & & & & & \hline & & & & & 66682 \end{array}$$

$$\begin{array}{r} 3420205 \\ 66682 \\ 3510 \\ 244 \\ 19 \end{array}$$

$$\begin{array}{l} \text{Length of arc} = \cdot 3490660 \\ \cdot 3490658 \text{ arc of } 20^\circ. \end{array}$$

When the sine of the required arc is greater than $\cdot 3$, perhaps it is better to find the sine of half the arc by the well-known formula of verification.

$$2 \sin \theta = \sqrt{1 + \sin 2\theta} - \sqrt{1 - \sin 2\theta}, \quad (\text{I.})$$

$$\text{suppose } \sin 2\theta = \cdot 3420205$$

$$\sqrt{1 + 3420205} = 1 \cdot 1584561$$

$$\sqrt{1 - 3420205} = \cdot 8111594$$

$$2 \cdot 3472967$$

$$\sin \theta = \cdot 1736484$$

$$\cdot 1736484 = 173 \downarrow 0,0,3,7,5,$$

$$\cdot 173 \times \cdot 173 = \cdot 0299290.$$

$$(\cdot 173)^3 = \cdot 0051777$$

$$(\cdot 173)^5 = \cdot 0001550$$

$$\frac{1}{2} \\ 51777$$

$$\begin{array}{r} 3 \\ 2588 \overline{)8} \\ 78 \dots \end{array} \sim 3$$

$$\begin{array}{r} 26121 \overline{)18} \dots \end{array} \sim 7$$

$$\begin{array}{r} 26175 \overline{)1} \dots \end{array} \sim 5$$

$$\downarrow 0,0,9,21,15,$$

$$\begin{array}{r} 862 \overline{)9} \dots \\ 7 \overline{)8} \dots \end{array}$$

$$\begin{array}{r} 8707 \overline{)18} \dots \end{array}$$

$$\begin{array}{r} 8725 \overline{)1} \dots \end{array}$$

$$8726$$

$$\frac{1}{2} \\ 1550$$

$$\frac{3}{4} \\ 775$$

$$\begin{array}{r} 5 \\ 581 \overline{)2} \dots \end{array} \sim 3$$

$$\downarrow 0,0,15,35,25,$$

$$\begin{array}{r} 116 \overline{)2} \dots \end{array}$$

$$118$$

$$\begin{array}{r} \cdot 1736484 \\ 8726 \\ \hline 118 \end{array}$$

$$\cdot 1745328 = \text{an arc of } 10^\circ \text{ very nearly.}$$

The seventh power of $(\cdot 173)$ would give $\cdot 0000002$, which may be added. If the sine of 2θ be given to find the cosine of θ , then the formula is

$$2 \cos \theta = \sqrt{1 + \sin 2\theta} + \sqrt{1 - \sin 2\theta}, \quad (\text{II.})$$

In the last example,

$$\sqrt{1.3420205} = 1.1584561$$

$$\sqrt{\cdot 6579795} = \cdot 8111594$$

$$2) 1.9696155$$

$$\cdot 9848078 = \cos 10^\circ = \sin 80^\circ.$$

To these formulæ may be added the following well-known expressions, namely,

$$\sin 2\theta = 2 \sin \theta \cos \theta; \quad (\text{III.})$$

$$\cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta, \quad (\text{IV.})$$

$$\cos \theta = \sqrt{\frac{1 + \cos 2\theta}{2}}, \quad (\text{V.})$$

$$\sin \theta = \sqrt{\frac{1 - \cos 2\theta}{2}}, \quad (\text{VI.})$$

13. *Given the cosine of an angle = .88888, find the degrees and minutes contained in the arc, working to five places of decimals by (VI.).*

$$\begin{array}{r} 1.00000 \\ .88888 \\ \hline 2) .11112 \end{array}$$

$$\sqrt{.05556} = .23571 = 23 \downarrow 0, 2, 4, 6, 3,$$

$$\begin{array}{r} 23 \overline{) 000} \\ \underline{460} \\ 2 \end{array}$$

$$\begin{array}{r} 234 \overline{) 62} \\ \underline{94} \dots \\ 14 \dots \dots \\ \underline{1} \dots \dots \end{array}$$

$$(.23)^2 = .05290$$

$$(.23)^3 = .01217$$

$$(.23)^5 = .00064$$

$$\frac{1}{2} \text{ of } 64$$

$$\frac{3}{4} \text{ of } 32$$

$$\frac{5}{24}$$

$$5$$

$\frac{1}{2}$ of 64 = 32; $\frac{3}{4}$ of 32 = 24, &c.; 5 operated upon by $\downarrow 0, 10, \dots$ is not increased a unit.

$$\frac{1}{2} \\ 1217$$

$$\begin{array}{r} 3 \\ 60 \overline{) 9} \\ 12 \cdot \end{array} \sim 2$$

$$\begin{array}{r} 645 \\ 3 \dots \overline{) 4} \end{array}$$

↓ 0,6,12,18, &c.

$$\begin{array}{r} 20 \overline{) 3} \cdot \\ 1 \overline{) 2} \cdot \end{array}$$

$$\begin{array}{r} 21 \overline{) 5} \dots \\ 3 \overline{) \dots} \\ \hline 21 \ 8 \end{array}$$

$$\cdot 23571$$

$$218$$

$$5$$

$$4) \cdot 237 \cdot 94 \text{ length of arc.}$$

$$59 \cdot 48$$

$$13) 178 \cdot 46$$

$$13 \cdot 7 \overline{) 3} \dots$$

$$8 \overline{) \dots} \sim 6$$

$$1 \overline{) \dots} \sim 9$$

$$13 \cdot 64$$

$$60$$

$$38 \cdot 40$$

$13^\circ 38'$ doubled = $27^\circ 16'$, the angle whose cosine = $\cdot 88888$.

If S be the seconds in an arc whose length is L , radius = 1,

Then,

$$S = L 200000 \times \downarrow 0,3,1,0,0,\overline{7},0,\overline{5},$$

and

$$L = \frac{4S}{1000000} \downarrow 2,0,2,\overline{3},\overline{2},0,7,.$$

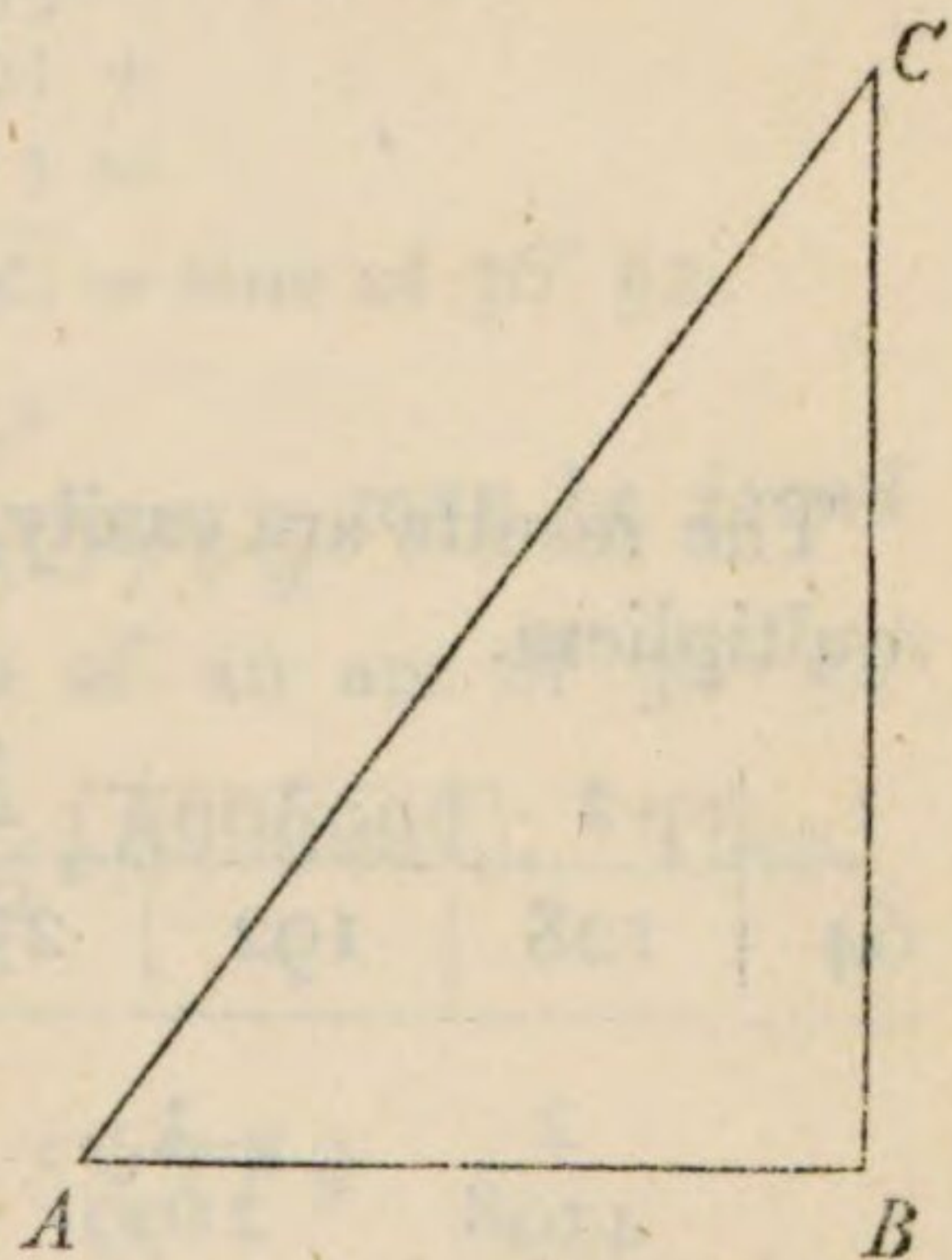
PART V.

FORMULÆ AND RULES FOR THE SOLUTION OF
PLANE TRIANGLES, WITHOUT THE USE OF
TABLES.*Examples.*

Fig. 1.

1. In a right-angled plane triangle ABC given the angle $CAB = 53^\circ 8'$, and the base $AB = 288$; find the perpendicular CB and the hypotenuse AC .

Since the angle $CAB +$ angle $ACB = 90^\circ$,



$$\begin{aligned}
 \therefore CAB &= \begin{array}{r} 90^\circ \text{ } 0' \text{ from} \\ 53 \quad 8 \text{ take} \\ \hline 36 \quad 52 \\ 60 \\ \hline \end{array} \\
 &= 2212 \text{ minutes.}
 \end{aligned}$$

Length of an arc of 1000' = .29089

" " 2000 = .58178

" " 3000 = .87267

" " 4000 = 1.16355

" " 5000 = 1.45444

" " 6000 = 1.74533

" " 7000 = 2.03622

" " 8000 = 2.32711

" " 9000 = 2.61799

2000' = .58178

200 = .05818

10 = .00291

2 = .00058

.64345 Length of an arc of 36° 52'.

.640 ↓ 0,0,5,3,7, = .64345

(.64)² = .40960

(.64)³ = .26214

(.64)⁴ = .16777

(.64)⁵ = .10737

(.64)⁶ = .06872

(.64)⁷ = .04398

The results are easily found by means of the following line of multipliers.

1	2	3	4	5	6	7	8	9
64	128	192	256	320	384	448	512	576

2	3	4	5	6	7	
4398	2199	733	184	37	6	1

2	3	4	5	6	
6872	3436	1145	286	57	10

2	3	4	5		↓ 0,0,25,15,35,
10737	5369	1789	447 .		89
			2 ~ 5		2
					91

$$\begin{array}{r}
 \begin{array}{c} 2 \\ 16777 \end{array} \quad \begin{array}{c} 3 \\ 8389 \end{array} \quad \begin{array}{c} 4 \\ 2796 \\ 14 \dots \end{array} \sim 5 \quad \downarrow 0,0,20,12,28, \\
 \begin{array}{c} 699 \\ 14 \end{array} \dots \quad \begin{array}{c} 713 \\ 1 \end{array} \dots \quad 714
 \end{array}$$

$$\begin{array}{r}
 \begin{array}{c} 2 \\ 26214 \end{array} \quad \begin{array}{c} 3 \\ 1310 \\ 66 \dots \end{array} \sim 5 \quad \downarrow 0,0,15,9,21, \\
 \begin{array}{c} 436 \\ 66 \\ 4 \end{array} \begin{array}{c} 9 \\ \dots \\ \dots \end{array} \quad 4439
 \end{array}$$

$$\begin{array}{r}
 \begin{array}{c} 2 \\ 40960 \end{array} \quad \downarrow 0,0,10,6,14, \\
 \begin{array}{c} 204 \\ 205 \\ 1 \end{array} \begin{array}{c} 80 \\ \dots \\ \dots \end{array} = A \\
 \begin{array}{c} 2068 \\ 12 \\ 3 \end{array} \begin{array}{c} 6 \\ \dots \\ \dots \end{array} = A \times 9 \div 2 \\
 20701
 \end{array}$$

$$\begin{array}{r}
 1.00000 + \\
 .20701 - \\
 714 + \\
 10 -
 \end{array}$$

$$.80013 = \text{sine of } 53^\circ 8'.$$

$$\begin{array}{r}
 .64345 + \\
 4439 - \\
 91 + \\
 1 -
 \end{array}$$

$$.59996 = \text{sine of } 36^\circ 52'.$$

The value of any term, as $+\frac{x^9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}$ may be found independently; for example, the length of an arc of $36^\circ 52'$ $= .643444724 = \frac{6}{10} \downarrow 0,7,0,2,5,4,1,6, = \frac{6}{10} \sqrt[6]{6990997}$. Then,

$$\begin{array}{r}
 6990997+ \quad - 396847197 \\
 6=179184951+ \quad -1280247082=1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \\
 186175948+ \quad -1677094279 \\
 10=230270081- \quad -1842160648=10^8 \\
 44094133- \quad 165066369=5 \downarrow 0,4,1,3,4,2,0,3,=5 \cdot 2100046. \\
 9 \\
 396847197-
 \end{array}$$

$$\therefore \frac{x^9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} = (.643444724)^9 \div 362880 = .000000052100046.$$

$$\text{As } \sin 36^\circ 52' : 288 :: \sin 90^\circ : AC.$$

$$\therefore .59996 : 288 :: 1.00000 : 480.036 = AC.$$

$$\text{Again, as } \sin 36^\circ 52' : 288 :: \sin 53^\circ 8' : CB.$$

$$\therefore .59996 : 288 :: .80003 : 384.045 = CB.$$

2. Given the two perpendicular sides to find the hypotenuse and angles, that is to say, $AB = 472$; $BC = 765$; find AC and the angles CAB , ACB .

$$\sqrt{472^2 + 765^2} = 898.8933;$$

$$\frac{472}{898.8933} = \sin ABC = .52498,$$

$$\sqrt{1.5249789} = 1.234901$$

$$\sqrt{.4750211} = .689218$$

$$2) .545683$$

$$.272842$$

$$.272 \downarrow 0,0,3,0,9, = .272842$$

$$\text{Square of } .272 = .073984$$

$$\text{Cube of } .272 = .020124$$

$$\text{Fifth power of } .272 = .001489$$

$$\text{Seventh power of } .272 = .000110$$

$$\frac{1}{2} \\ 110$$

$$\frac{3}{4} \\ 55$$

$$\frac{5}{6} \\ 42$$

$$7 \\ 35$$

$$5$$

$$\frac{1}{2} \\ 1489$$

$$\frac{3}{4} \\ 745$$

$$\begin{array}{r} 5 \\ 559 \\ 2 \dots \end{array} \sim 3$$

$$\downarrow 0,0,15,0,45,$$

$$\begin{array}{r} 112 \\ 2 \end{array} \dots$$

$$114$$

$$\frac{1}{2} \\ 20124$$

$$3 \\ 10062$$

$$\downarrow 0,0,9,0,27,$$

$$\begin{array}{r} 335 \\ 3 \end{array} \dots$$

$$\begin{array}{r} 3384 \\ 1 \end{array} \dots$$

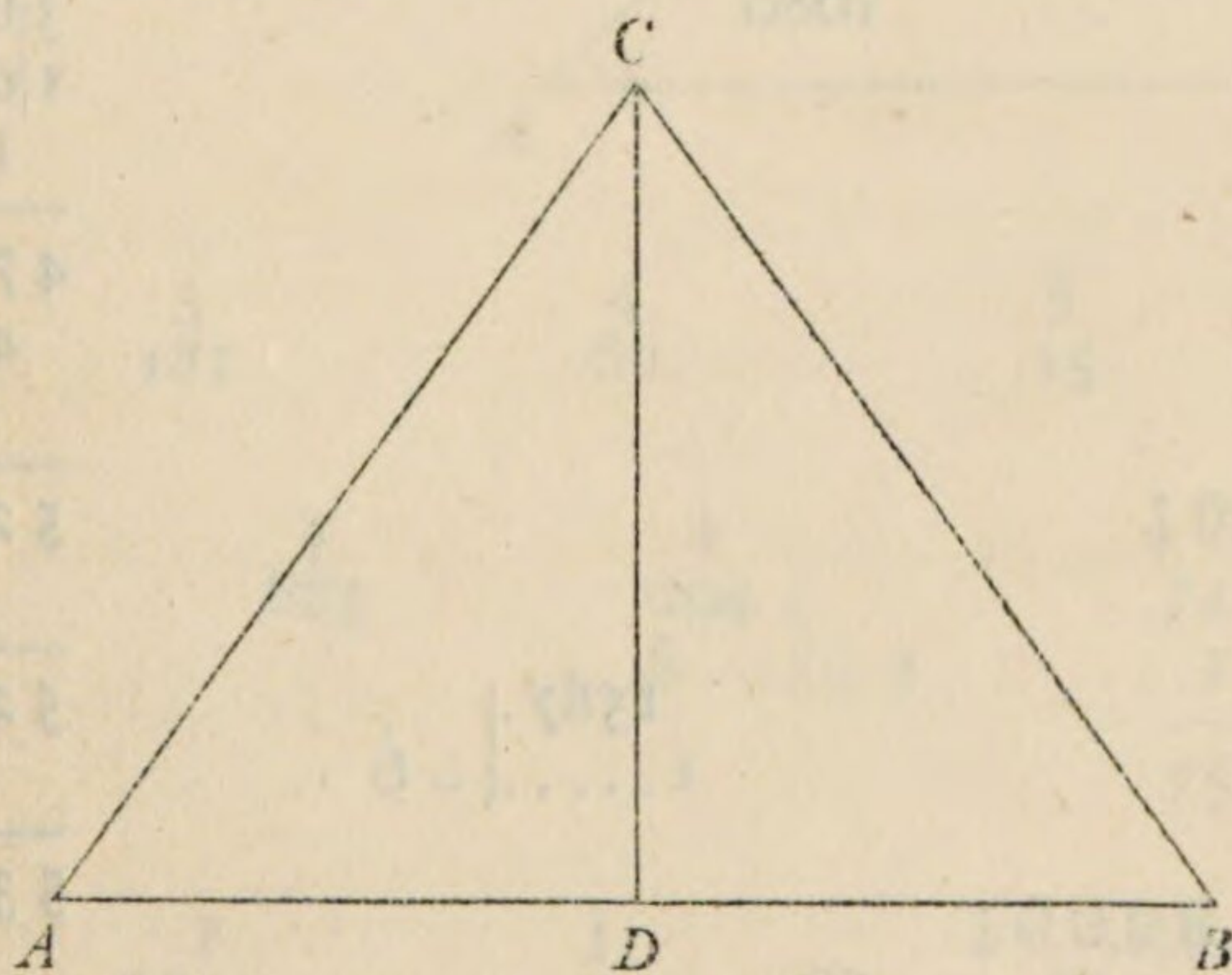
$$3385$$

$$\begin{array}{r}
 .272842 \\
 3385 \\
 114 \\
 5 \\
 \hline
 .276346 \text{ length of arc.} \\
 900' = .261799 \\
 \hline
 .014547 \\
 50' = .014544 \\
 \hline
 950' = 15^\circ 50' \\
 \therefore 31\ 40 = \text{angle } ACB.
 \end{array}$$

$$\begin{array}{r}
 \text{From } 90^\circ\ 0' \\
 \text{take } 31\ 40 \\
 \hline
 58^\circ\ 20' = \text{angle } CAB.
 \end{array}$$

If the angles be required to seconds, the decimals must extend to seven places.

Fig. 2.



3. Given two sides and the included angle of an isosceles triangle ACB , to find the other parts. $AC = CB = 288$, $ACB = 78^\circ 12'$.

Draw CD perpendicular to AB , then

$$ADC = 90^\circ \quad 0'$$

$$ACD = 39 \quad 6$$

$$CAD = 50^\circ 54' = CBD.$$

The work of this example is only extended to four decimal places.

$$39^\circ 6' = 2346', \text{ length of arc} = \cdot 6824.$$

$$6 \downarrow 1,3,3,6, = \cdot 6824$$

$$(\cdot 6)^3 = \cdot 2160$$

$$(\cdot 6)^5 = \cdot 0778$$

$$\begin{array}{r} 2 \\ 778 \end{array}$$

$$\begin{array}{r} 3 \\ 389 \end{array}$$

$$\begin{array}{r} 4 \\ 130 \end{array}$$

$$\begin{array}{r} 5 \\ 33 \end{array}$$

$$\downarrow 5,15,15,30$$

$$\begin{array}{r} 7| \\ 4| \end{array}$$

$$\begin{array}{r} 11| \dots | \dots \\ 2| \dots | \dots \end{array}$$

$$13$$

$$\begin{array}{r} 2 \\ 2160 \end{array}$$

$$\begin{array}{r} 3 \\ 1080 \end{array}$$

$$\downarrow 3,9,9,18,$$

$$\begin{array}{r} 3|6|0| \\ 1|0|8| \\ 1|1| \end{array}$$

$$\begin{array}{r} 47|9 \cdot \\ 43 \cdot \\ 2 \cdot \end{array}$$

$$\begin{array}{r} 524| \dots \\ 5| \dots \end{array}$$

$$\begin{array}{r} 529| \dots \dots \\ 1| \dots \dots \end{array}$$

$$530$$

$$\begin{array}{r} 1587| \\ 1 \dots \dots | \sim 6 \end{array}$$

$$6824 +$$

$$530 -$$

$$\hline 6294$$

$$13 +$$

$$\hline \cdot 6307 = \sin 39^\circ 6'.$$

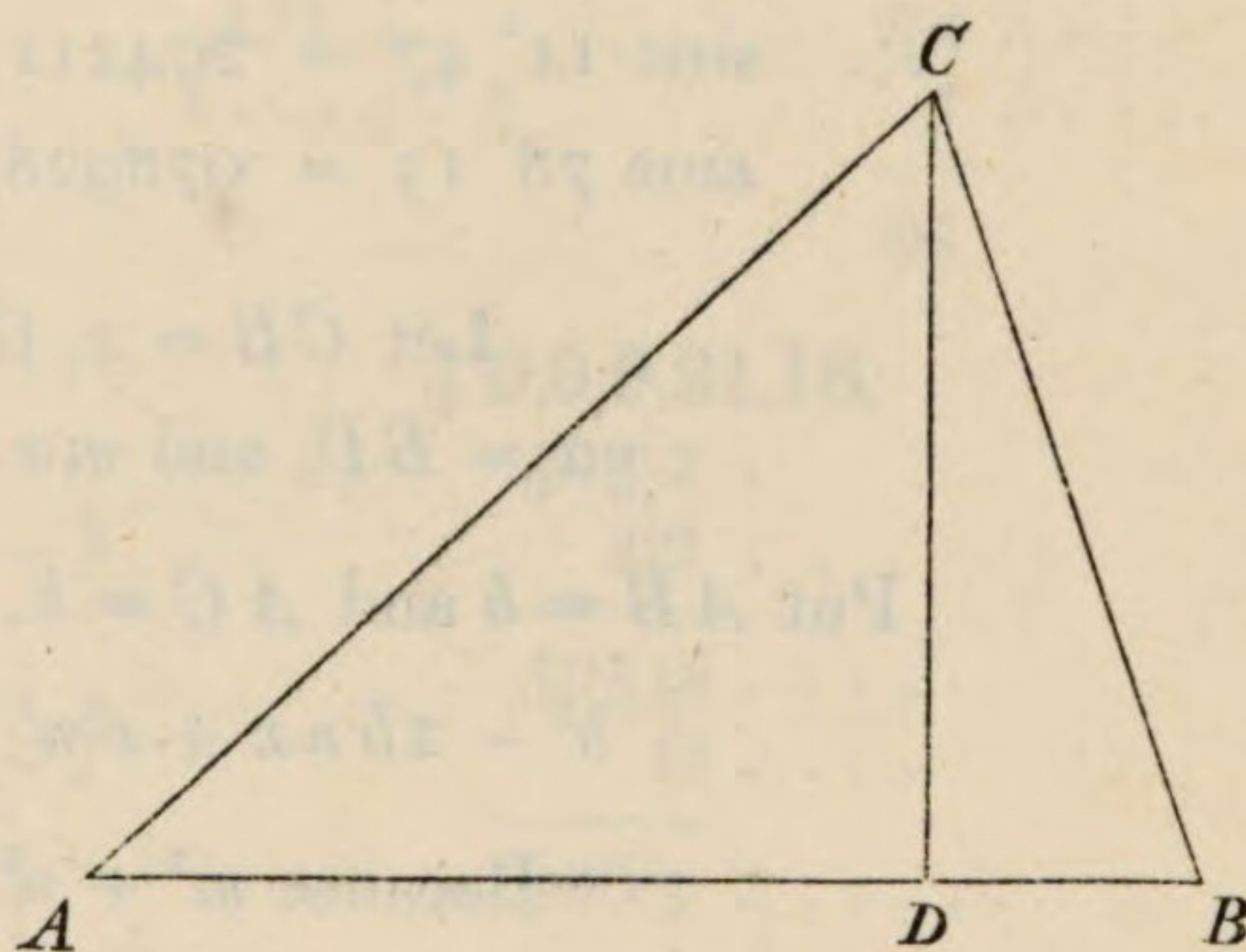
$$\begin{aligned}\sin ADC : AC &:: \sin ACD : AD, \\ \text{or as } \sin 90^\circ : 288 &:: .6307 : AD, \\ \therefore 1 : 288 &:: .6307 : 181.64.\end{aligned}$$

$$AD = 181.64 \quad \therefore AB = 363.28.$$

4. In the triangle ABC are given $AB = 137$, $AC = 153$, and angle $CBA = 78^\circ 13'$, to find the other parts.

Draw CD perpendicular to AB , then $DCB = 11^\circ 47' = 707'$; arc of $707' = .205658$.

Fig. 3.



$$205 \downarrow 0,0,3,2, = 205658.$$

$$(.205)^2 = .042025$$

$$(.205)^3 = .008615$$

$$(.205)^4 = .001766$$

$$(.205)^5 = .000362$$

$$\begin{array}{r} 2 \\ 362 \end{array}$$

$$\begin{array}{r} 3 \\ 181 \end{array}$$

$$\begin{array}{r} 4 \\ 60 \end{array}$$

$$\begin{array}{r} 5 \\ 15 \end{array}$$

$$3$$

$$\begin{array}{r} 2 \\ 1766 \end{array}$$

$$\begin{array}{r} 3 \\ 883 \end{array}$$

$$\begin{array}{r} 4 \\ 294 \cdot \\ 8 \dots \end{array} \Big| \sim 3$$

$$\begin{array}{r} \downarrow 0,0,12,8, \\ 74 \cdot \dots \\ 1 \cdot \dots \\ \hline 75 \end{array}$$

$$\begin{array}{r} 2 \\ 8615 \end{array}$$

$$\begin{array}{r} 3 \\ 430 \cdot 8 \\ 13 \dots \end{array} \Big| \sim 3$$

$$\begin{array}{r} \downarrow 0,0,9,6, \\ 143 \cdot 6 \dots \\ 1 \cdot 3 \dots \\ \hline 1449 \dots \\ 1 \dots \\ \hline 1450 \end{array}$$

$$\begin{array}{r} 2 \\ 420 \overline{) 25} \\ 126 \cdot \end{array} \sim 3$$

$$\begin{array}{r} \downarrow 0,0,6,4, \\ 210 \overline{) 13} \cdot \\ 1 \overline{) 26} \cdot \\ \hline 2113 \overline{) 9} \dots \\ 8 \dots \\ \hline 21147 \end{array}$$

$$\begin{array}{r} 1 \cdot 000000 \\ \cdot 021147 - \\ \hline \cdot 978853 \\ 75 + \\ \hline \cdot 978928 \end{array}$$

$$\begin{array}{r} \cdot 205658 \\ 1450 - \\ \hline \cdot 204208 \\ 3 + \\ \hline \cdot 204211 \end{array}$$

$$\begin{aligned} \therefore \sin 11^\circ 47' &= \cdot 204211, \text{ which put } = n, \\ \sin 78 \ 13 &= \cdot 978928, \quad \quad \quad = m. \end{aligned}$$

Let $CB = x$, then

$$nx = BD, \text{ and } mx = CD.$$

Put $AB = b$ and $AC = h$. $AD = b - nx$.

$$\therefore b^2 - 2bnx + x^2n^2 + x^2m^2 = h^2.$$

Because $m^2 + n^2 = 1$,

$$x = nb \pm \sqrt{(h + mb)(h - mb)}, = 101 \cdot 616.$$

$$153 : \cdot 978928 :: 101 \cdot 616 : \sin CAB = \cdot 650161.$$

Again, $\sin ABC : AC :: \sin ACB : AB$,

$$\therefore \frac{137 \times \cdot 978928}{153} = \sin ACB = \cdot 876556.$$

To find the sine of half this angle :

$$\sqrt{1 \cdot 876556} = 1 \cdot 369875$$

$$\sqrt{\cdot 123446} = \cdot 351336$$

$$2) 1 \cdot 018529$$

$$\text{sine of half } ACB = \cdot 509265$$

$$\sqrt{1.509265} = 1.228522$$

$$\sqrt{.490735} = .700525$$

$$2) \cdot 527997$$

$$\sin \text{ of } \frac{1}{4} \text{ of } ACB = .263998$$

$$263 \downarrow 0,0,3,7,6, = 263998$$

$$(.263)^3 = .018191$$

$$(.263)^5 = .001258$$

$$(.263)^7 = .000087$$

$\frac{1}{2}$ 87	$\frac{3}{4}$ 44	$\frac{5}{6}$ 33	$\frac{7}{28}$	4
$\frac{1}{2}$ 1258	$\frac{3}{4}$ 629	5 472 . 1 ~ 3	$\downarrow 0,0,15,35,30,$ 94 1 95	

$\frac{1}{2}$ 18191	3 909 6 27 . . ~ 3	$\downarrow 0,0,9,21,18,$ 303 2 . . 2 7 . . 305 9 6 306 5 1 306 6
	9177 6 ~ 7 9195 . 5 ~ 6	

$$263998$$

$$3066$$

$$95$$

$$4$$

Length of arc = $.267163$, and will be found to correspond to $918' 27'' = 15^\circ 18' 27''$, which, when multiplied by 4, gives $61^\circ 13' 48''$ for the angle ACB .

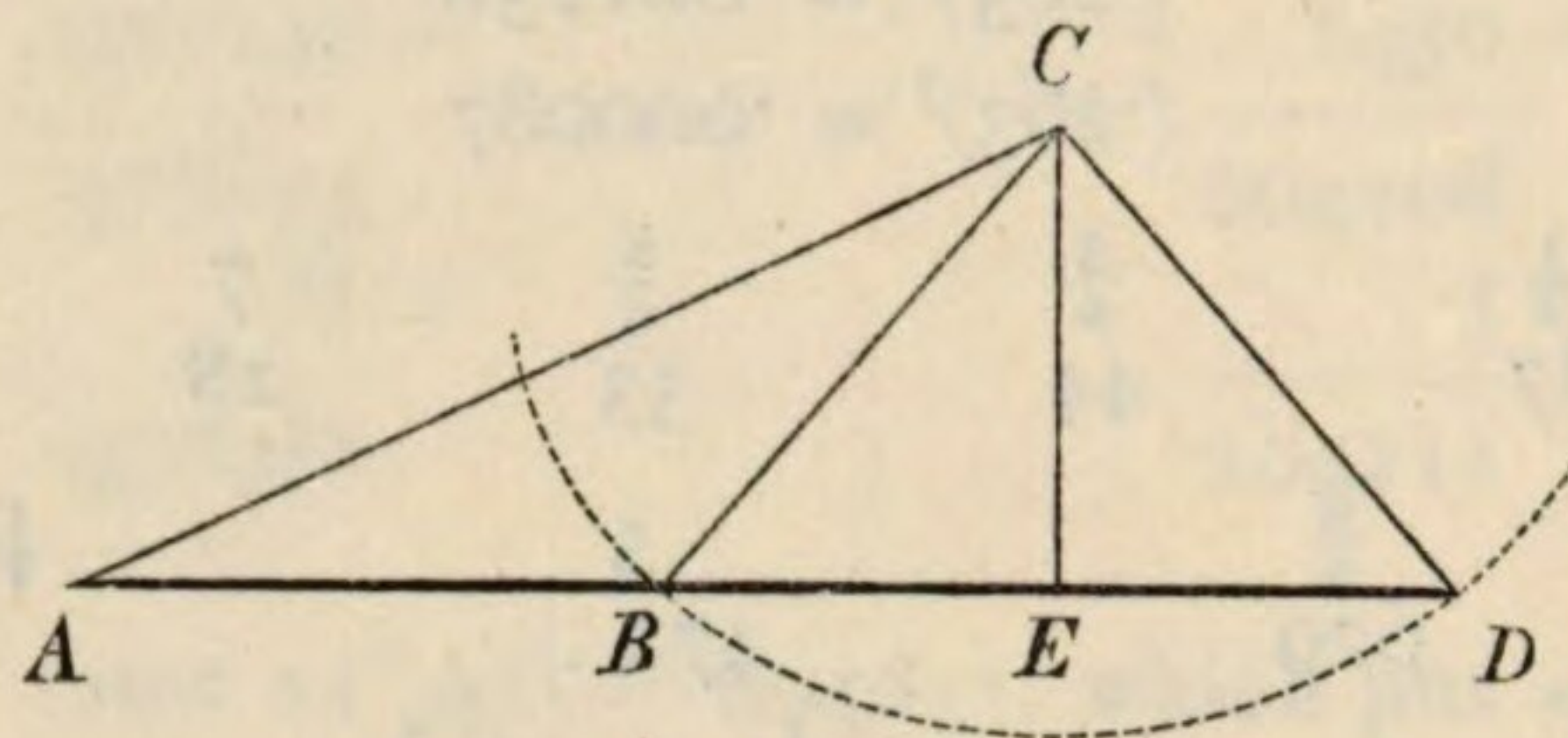
61° 13' 48"
78 13 0
Take 139° 26' 48"
from 180 0 0

$$40^\circ 33' 12'' = \text{angle } CAB.$$

5. In the plane triangle ABC or ACD , are given $AC = 216$, CB or CD 117, angle $CAB = 22^\circ 37'$, to find the remaining parts.

Let $AC = b$, CB or $CD = a$, $AB = n$, $AD = m$, and angle $CAB = \theta$.

Fig. 4.



$$\text{Then, } CE = b \sin \theta$$

$$AE = b \cos \theta$$

$$BE^2 = ED^2 = a^2 - b^2 \sin^2 \theta$$

$$\therefore AB = n = b \cos \theta - \sqrt{a^2 - b^2 \sin^2 \theta}$$

$$AD = m = b \cos \theta + \sqrt{a^2 - b^2 \sin^2 \theta}$$

$$(a + b \sin \theta)(a - b \sin \theta) = a^2 - b^2 \sin^2 \theta.$$

$$22^\circ 37' = 1357', \text{ length of this arc} = \cdot 394734$$

$$394 \downarrow 0,0,1,8,6, = 394734$$

$$(\cdot 394)^2 = \cdot 155236$$

$$(\cdot 394)^3 = \cdot 061163$$

$$(\cdot 394)^4 = \cdot 024098$$

$$(\cdot 394)^5 = \cdot 009495$$

$$(\cdot 394)^6 = \cdot 003741$$

2
9495

3
4748

4
1583

5
396

79

$$\begin{array}{r} 2 \\ 61163 \end{array}$$

$$\begin{array}{r} 3 \\ 30582 \end{array}$$

$$\downarrow 0,0,3,24,18,$$

$$\begin{array}{r} 101 \overline{) 94} . \\ 31 . \end{array}$$

$$\begin{array}{r} 3067 \overline{) 5} \\ 25 \dots \end{array} \sim 8$$

$$\begin{array}{r} 102 \overline{) 25} \dots \\ 2 \overline{) 5} \dots \end{array}$$

$$\begin{array}{r} 30750 \overline{) 2} \dots \dots \dots \\ 2 \dots \dots \end{array} \sim 6$$

$$\begin{array}{r} 102 \overline{) 50} \dots \dots \dots \\ 2 \overline{) 0} \dots \dots \end{array}$$

$$102 \ 5 \ 2$$

$$\begin{array}{r} .394734 + \\ 10252 - \end{array}$$

$$.384482$$

$$79 +$$

$$.384561 = \sin 22^\circ 37'.$$

$$216 \times .384561 = 83.065.$$

$$117.000$$

$$83.065$$

$$33.935 \text{ diff.}$$

$$200.065 \text{ sum.}$$

$$\sqrt{200.065 \times 33.935} = 82.396.$$

To find the cosine of $22^\circ 37'$.

$$\begin{array}{r} 2 \\ 3741 \end{array}$$

$$\begin{array}{r} 3 \\ 1872 \end{array}$$

$$\begin{array}{r} 4 \\ 624 \end{array}$$

$$\begin{array}{r} 5 \\ 156 \end{array}$$

$$\begin{array}{r} 6 \\ 31 \end{array}$$

$$5$$

$$\begin{array}{r} 2 \\ 24098 \end{array}$$

$$\begin{array}{r} 3 \\ 12049 \end{array}$$

$$\begin{array}{r} 4 \\ 4016 \end{array}$$

$$\downarrow 0,0,4,32,24,$$

$$\begin{array}{r} 100 \overline{) 4} \dots \\ 4 \dots \end{array}$$

$$\begin{array}{r} 4032 \overline{) 3} \dots \dots \dots \\ 3 \dots \dots \end{array} \sim 8$$

$$\begin{array}{r} 100 \overline{) 8} \dots \dots \dots \\ 3 \overline{) 0} \dots \dots \end{array}$$

$$101 \ 1$$

$$0$$

$$\begin{array}{r} 2 \\ 155236 \end{array}$$

$$\begin{array}{r} 15554 \overline{)6} \\ 124 \dots \end{array} \sim 8$$

$$\begin{array}{r} 155794 \overline{)6} \\ 9 \dots \end{array} \sim 6$$

$$\downarrow 0,0,2,16,12,$$

$$\begin{array}{r} 776 \overline{)18} \\ 1 \overline{)55} \end{array}$$

$$\begin{array}{r} 7777 \overline{)3} \dots \\ 12 \overline{)4} \dots \end{array}$$

$$\begin{array}{r} 77897 \overline{) \dots} \\ 9 \overline{) \dots} \end{array}$$

$$77906$$

$$\begin{array}{r} 1.000000 + \\ 77906 - \end{array}$$

$$\cdot 922094$$

$$1011 +$$

$$\cdot 923105$$

$$5 -$$

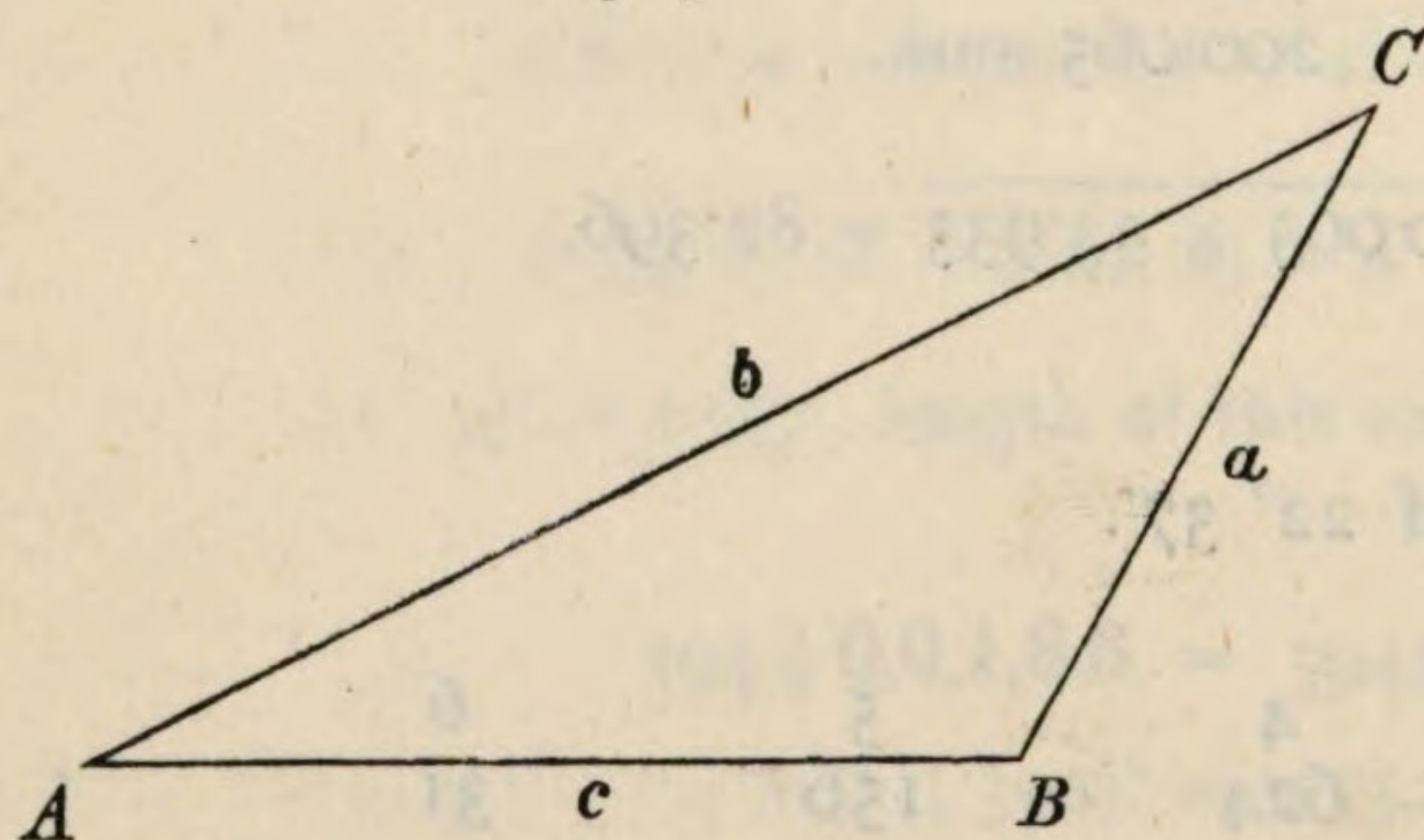
$$\cdot 923100 = \cos 22^\circ 37'.$$

$$\begin{array}{r} 216 \times \cdot 9231 = 199.3896 \\ 82.396 \end{array}$$

$$281.7856 \text{ sum} = AD,$$

$$116.9936 \text{ diff.} = AB.$$

Fig. 5.



Let ABC be any plane triangle, and denote the angles by the letters A, B, C , at their vertices, and the sides opposite to them by the small letters a, b, c ; represent $a + b + c$ by $2s$.

Then

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}};$$

$$\sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}};$$

$$\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}};$$

$$\begin{array}{r} 116.994 \\ 216.000 \\ 117.000 \\ \hline \end{array}$$

$$2 \) \ 449.994$$

$$224.997 = s$$

$$\begin{array}{r} 224.997 \\ 117.000 \\ \hline \end{array}$$

$$107.997 = s - a$$

$$\begin{array}{r} 224.997 \\ 216.000 \\ \hline \end{array}$$

$$8.997 = s - b.$$

$$\sqrt{\frac{107.997 \times 8.997}{117 \times 216}} = .196081.$$

$$196 \downarrow 0,0,0,4,2, = 196081$$

$$(.196)^3 = .007530$$

$$(.196)^5 = .000289$$

$$\frac{1}{2} \begin{array}{r} 289 \end{array}$$

$$\frac{3}{4} \begin{array}{r} 144 \end{array}$$

$$\frac{5}{108}$$

$$22$$

$$\frac{1}{2} \begin{array}{r} 7530 \end{array}$$

$$\begin{array}{r} 3 \\ 3765 \\ 2 \dots \mid \sim 4 \end{array}$$

$$\downarrow 0,0,0,12,6,$$

$$\begin{array}{r} 1255 \mid \dots \\ 2 \mid \dots \\ \hline \end{array}$$

$$1257$$

$$.196081$$

$$1257$$

$$22$$

$$.197360 = \text{arc of } 678' 28''.6 = 11^\circ 18' 28''.6,$$

the double of which is $22^\circ 36' 57''.2 = ACB$.

$$\begin{array}{r} 22^\circ \downarrow 36' 57'' \\ 22 \quad 37 \quad 0 \end{array}$$

$$\begin{array}{l} \text{Take} \\ \text{from} \end{array} \begin{array}{r} 45^\circ 13' 57'' \\ 180 \quad 0 \quad 0 \end{array} = CDE \text{ (fig. 4.)}$$

$$\begin{array}{r} 134^\circ 46' 3'' \\ \hline \end{array} = ABC$$

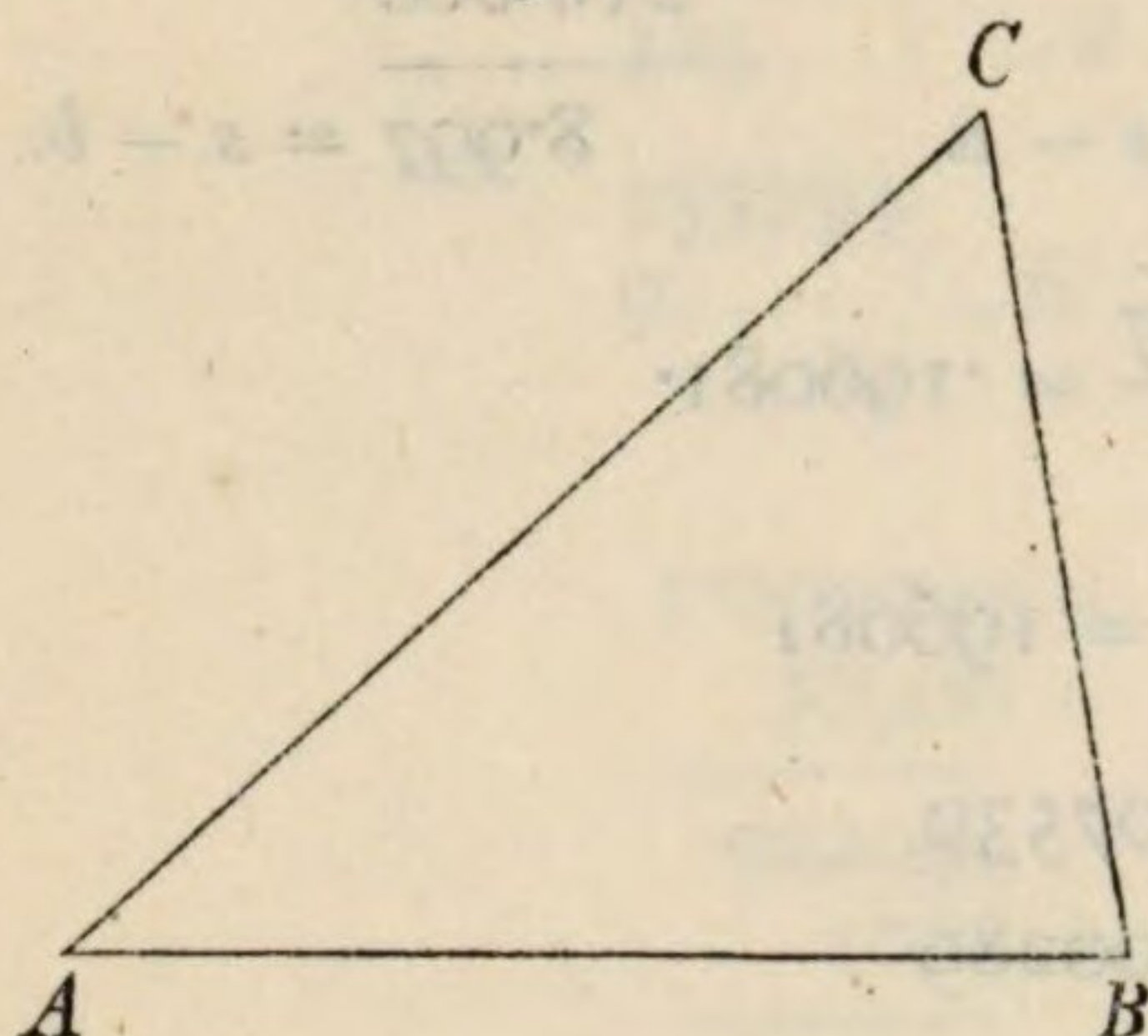
$$\begin{array}{r} 45^\circ 13' 57'' \\ 22 \quad 37 \quad 0 \end{array}$$

$$\begin{array}{l} \text{Take} \\ \text{from} \end{array} \begin{array}{r} 67^\circ 50' 57'' \\ 180 \quad 0 \quad 0 \end{array}$$

$$\begin{array}{r} 112^\circ 9' 3'' \\ \hline \end{array} = ACD \text{ (fig. 4.)}$$

6. In the plane triangle ABC , are given $AB = 408$ yards, $B = 74^\circ 14'$, $A = 49^\circ 23'$; to find the other sides.

Fig. 6.



As the three angles taken together make 180° ,

$$\therefore C = 56^\circ 23'.$$

$$\sin 56^\circ 23' = \cosine 33^\circ 37'.$$

$$\text{Length of arc of } 33^\circ 37' = \cdot 586721.$$

$$586 \downarrow 0,0,1,2,3, = 586721.$$

$$(\cdot 586)^2 = \cdot 343396$$

$$(\cdot 586)^4 = \cdot 117921$$

$$(\cdot 586)^6 = \cdot 040494$$

$$\begin{array}{r} 2 \\ 40494 \end{array}$$

$$\begin{array}{r} 3 \\ 20247 \end{array}$$

$$\begin{array}{r} 4 \\ 6749 \end{array}$$

$$\begin{array}{r} 5 \\ 1687 \end{array}$$

$$\begin{array}{r} 6 \\ 337 \end{array}$$

$$56$$

$$\begin{array}{r} 2 \\ 117921 \end{array}$$

$$\begin{array}{r} 3 \\ 58960 \end{array}$$

$$\begin{array}{r} 4 \\ 19653 \end{array}$$

$$\downarrow 0,0,4,8,12,$$

$$\begin{array}{r} 491 \overline{) 3 \dots} \\ 20 \dots \end{array}$$

$$\begin{array}{r} 493 \overline{) 3 \dots} \\ 4 \dots \end{array}$$

$$\begin{array}{r} 493 \overline{) 7 \dots} \\ 1 \dots \end{array}$$

$$493 \overline{) 8}$$

$$\begin{array}{r} 19748 \overline{) 6 \dots} \\ 6 \dots \end{array}$$

$$\begin{array}{r} 2 \\ 343396 \end{array}$$

$$\downarrow 0,0,2,4,8,$$

$$\begin{array}{r} 171 \overline{) 698 \dots} \\ 343 \dots \end{array}$$

$$\begin{array}{r} 172 \overline{) 041 \dots} \\ 69 \dots \end{array}$$

$$\begin{array}{r} 172 \overline{) 110 \dots} \\ 10 \dots \end{array}$$

$$172 \overline{) 120}$$

$$\begin{array}{r}
 1.000000 + \\
 172120 - \\
 \hline
 827880 \\
 4938 + \\
 \hline
 832818 \\
 56 - \\
 \hline
 \end{array}$$

$$\sin 56^\circ 23' = .832762 = \cos 33^\circ 37'.$$

$$\text{Length of arc of } 15^\circ 46' = .275180.$$

$$275 \downarrow 0,0,0,6,6, = 275180$$

$$(.275)^2 = .075625$$

$$(.275)^4 = .005719$$

$$\begin{array}{r}
 2 \\
 5719
 \end{array}$$

$$\begin{array}{r}
 3 \\
 2860
 \end{array}$$

$$\begin{array}{r}
 4 \\
 953 \cdot | \\
 6 \dots | \sim 6 -
 \end{array}$$

$$\begin{array}{r}
 \downarrow 0,0,0,24,24, \\
 238 \cdot | \dots \\
 1 \cdot | \dots \\
 \hline
 239
 \end{array}$$

$$\begin{array}{r}
 2 \\
 7562 | 5 \\
 45 \dots | \sim 6 -
 \end{array}$$

$$\begin{array}{r}
 75716 | \\
 45 \dots | \sim 6 -
 \end{array}$$

$$\downarrow 0,0,0,12,12,$$

$$\begin{array}{r}
 3781 | 3 \dots \\
 4 | 5 \dots
 \end{array}$$

$$\begin{array}{r}
 37858 | \dots \\
 5 | \dots
 \end{array}$$

$$37863$$

$$\begin{array}{r}
 1.000000 + \\
 37863 - \\
 \hline
 \end{array}$$

$$962137$$

$$239 +$$

$$\sin 74^\circ 14' = .962376 = \cos 15^\circ 46'.$$

$$90^\circ 0'$$

$$49 13$$

$$\text{Length of arc of } 40^\circ 47' = .711803$$

$$711 \downarrow 0,0,1,1,3, = .711803$$

$$(.711)^2 = .505521$$

$$(.711)^4 = .255551$$

$$(.711)^6 = .129186$$

$$(.711)^8 = .065306$$

$$\begin{array}{ccccccc} 2) & 3) & 4) & 5) & 6) & 7) & 8) \\ 65306 & 32653 & 10884 & 2721 & 544 & 91 & 13 \quad 2 \end{array}$$

$$\begin{array}{cccccc} 2) & 3) & 4) & 5) & 6) & \\ 129186 & 64593 & 21531 & 5383 & 1077 & \end{array} \quad \begin{array}{l} \downarrow 0,0,6,6,18, \\ 179 \mid \dots \\ \underline{1} \mid \dots \\ 180 \end{array}$$

$$\begin{array}{ccc} 2) & 3) & 4) \\ 255551 & 127776 & 42592 \end{array} \quad \begin{array}{l} \downarrow 0,0,4,4,12, \\ 106 \mid 48 \dots \\ \quad 43 \dots \\ 1069 \mid 1 \dots \\ \quad 4 \dots \\ \hline 10695 \mid \dots \\ \quad 1 \mid \dots \\ \hline 10696 \end{array}$$

$$\begin{array}{c} 2) \\ 505521 \end{array} \quad \begin{array}{l} \downarrow 0,0,2,2,6, \\ 252 \mid 762 \dots \\ \quad 506 \dots \\ \hline 2532 \mid 68 \dots \\ \quad 5 \mid 1 \dots \\ \quad 15 \dots \\ \hline 253334 \end{array}$$

$$\begin{array}{r} 1.000000 + \\ - 253334 - \\ \hline .746666 \\ + 10696 + \\ \hline .757362 \\ - 180 - \\ \hline .757182 \\ + 2 + \\ \hline \end{array}$$

$$\sin 49^\circ 13' = .757184 = \cos 40^\circ 47':$$

$$\sin 56^\circ 23' : 408 :: \sin 74^\circ 14' : AC.$$

$$\therefore .832762 : 408 :: .962376 : 471.5 = AC$$

$$\therefore .832762 : 408 :: .757184 : 371.9 = BC.$$

Fig. 7.

7. Given the two sides ($a = 562$, $b = 320$) and the included angle ($C = 128^\circ 4'$), to find the third side (c), and the remaining angles (B , A).

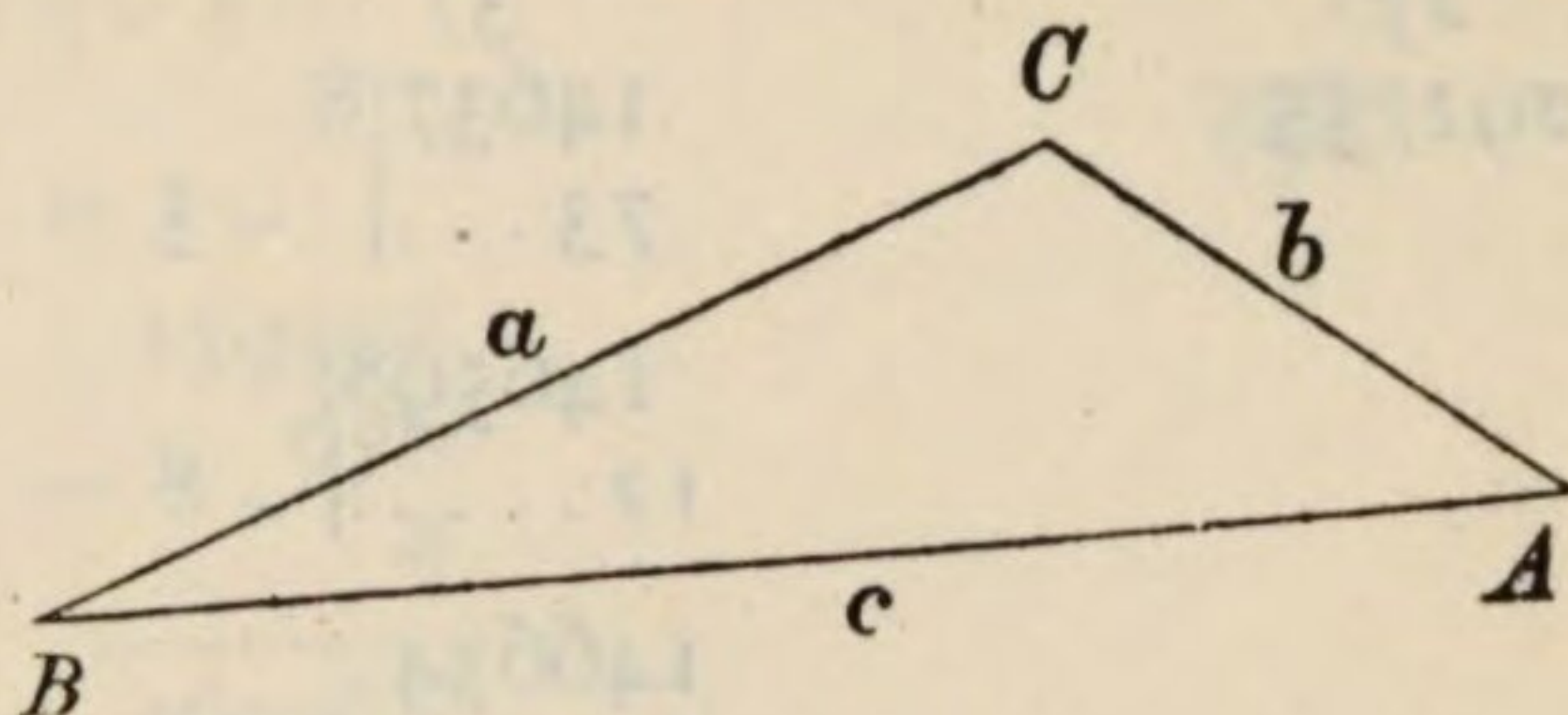
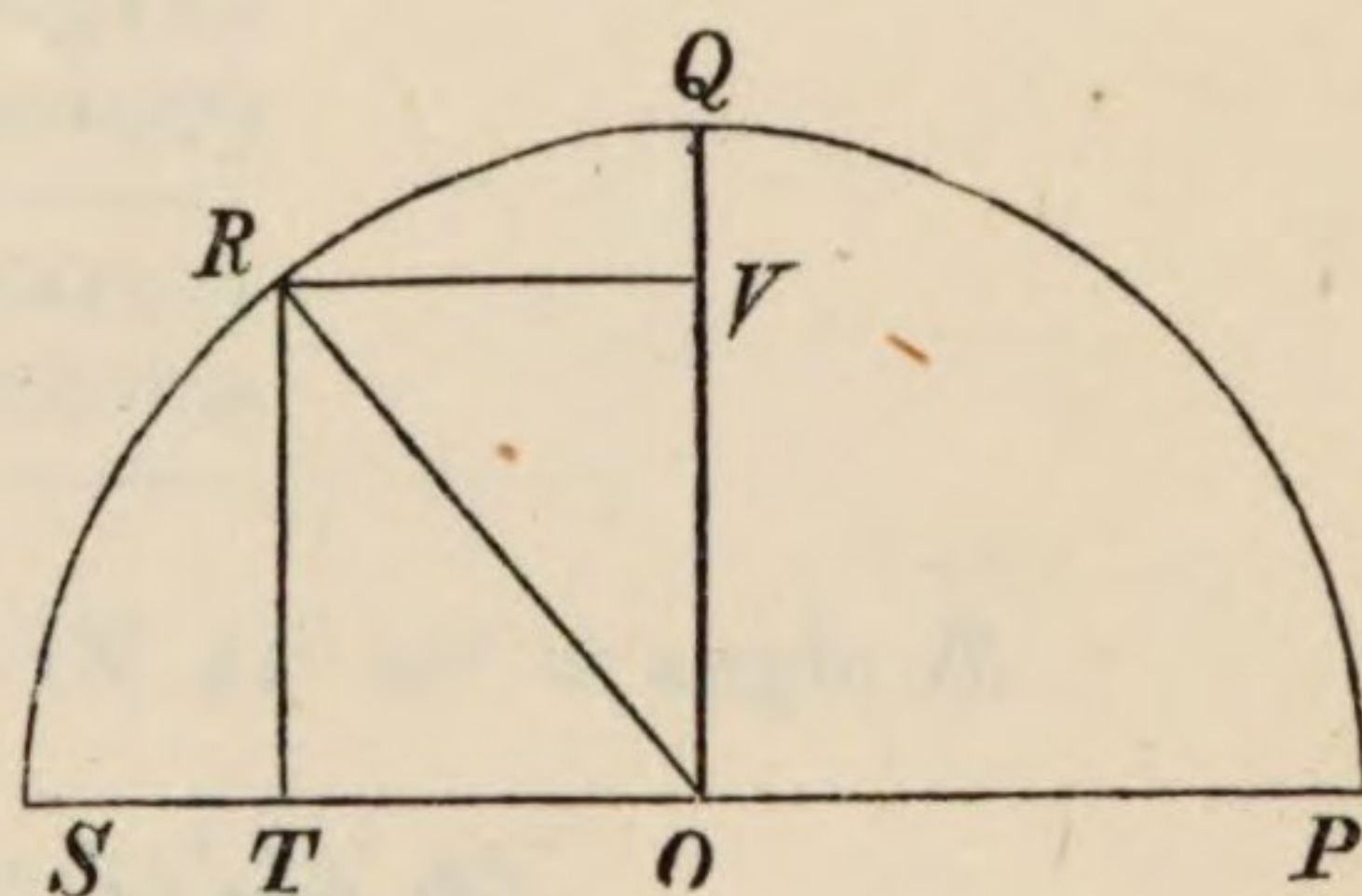


Fig. 8.

Let OP (fig. 8) be radius $= 1$, and the arc PQR one of $128^\circ 4'$, the cosine of this arc is $= TO = VR$, and is negative. But VR is the sine of the arc $QR = 128^\circ 4' - 90^\circ 0' = 38^\circ 4' = 2284'$; the length of this arc $= .664389$.



The well-known formula,

$$c^2 = a^2 + b^2 - 2ab \cos C \text{ (fig. 7),}$$

in this example becomes

$$c^2 = a^2 + b^2 + 2ab \cos C.$$

$$\therefore c^2 = (562)^2 + (320)^2 + 2 \times 320 \times 562 \times \cos C.$$

Now c is readily found when $\cos C$ becomes known; the length of the arc QR is given $= .664389$ to find RV .

$$664 \downarrow 0,0,0,5,8,6, = 664389$$

$$(664)^3 = .292755$$

$$(664)^5 = .129075$$

$$(664)^7 = .037787$$

2)	3)	4)	5)	6)	7)	
37787	18894	6298	1575	315	53	8

$$\begin{array}{r}
 2) \quad 129075 \\
 3) \quad 64538 \\
 4) \quad 21513 \\
 5) \quad 5378 \mid 27 \dots \mid \sim 5 - \\
 \downarrow 0,0,0,25,40,30, \\
 1076 \mid \dots \dots \dots \\
 3 \mid \dots \dots \dots \\
 \hline
 1079
 \end{array}$$

$$\begin{array}{r}
 2) \quad 292755 \\
 3) \quad 14637 \mid 8 \\
 73 \dots \mid \sim 5 - \\
 146598 \mid 12 \dots \dots \mid \sim 8 - \\
 146634 \mid 8 \dots \dots \mid \sim 6 - \\
 \downarrow 0,0,0,15,24,18, \\
 4879 \mid 3 \dots \dots \\
 7 \mid 3 \dots \dots \\
 \hline
 48866 \mid \dots \dots \dots \\
 12 \mid \dots \dots \dots \\
 \hline
 48878 \mid \dots \dots \dots \\
 1 \mid \dots \dots \dots \\
 \hline
 48879
 \end{array}$$

$$\begin{array}{r}
 664389 + \\
 48879 - \\
 \hline
 615510 \\
 1079 + \\
 \hline
 616589 \\
 8 - \\
 \hline
 616581
 \end{array}$$

$$\sin 38^\circ 4' = \cdot 616581$$

$$\therefore \cos 128^\circ 4' = \text{negative } \cdot 616581.$$

$$\therefore c^2 = 562^2 + 320^2 + 2 \times 320 \times 562 \times \cdot 616581$$

$$\therefore c = 800\cdot009 = AB.$$

To find B , the lesser of the remaining angles :

$$\sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$$

$$\therefore \sin \frac{B}{2} = \sqrt{\frac{279 \times 41}{562 \times 800}} = \cdot 15908$$

To find the arc and angle corresponding to this sine :

$$159 \downarrow 0,0,3,2, = \cdot 159508$$

$$(\cdot 159)^2 = \cdot 025281$$

$$(\cdot 159)^3 = \cdot 004020$$

$$(\cdot 159)^5 = \cdot 000102$$

$$\begin{array}{r}
 \frac{1}{2} \\
 102
 \end{array}
 \qquad
 \begin{array}{r}
 \frac{3}{4} \\
 51
 \end{array}
 \qquad
 \begin{array}{r}
 5) \\
 38
 \end{array}
 \qquad
 \begin{array}{r}
 8
 \end{array}$$

$$\begin{array}{r}
 \frac{1}{2} \\
 4020
 \end{array}
 \qquad
 \begin{array}{r}
 3) \\
 2010 \\
 6 \dots \sim 3 \sim
 \end{array}
 \qquad
 \begin{array}{r}
 \downarrow 0,0,9,6, \\
 670 \dots \\
 6 \dots \\
 \hline
 676
 \end{array}$$

$$\begin{array}{r}
 159508 \\
 676 \\
 8 \\
 \hline
 160192 \\
 9^\circ \sim 157080 \\
 \hline
 003112 \\
 10 \sim 002909 \\
 \hline
 000203 \\
 40'' \sim 000194 \\
 \hline
 \end{array}$$

$9^\circ 10' 40''$ doubled gives $18^\circ 21' 20'' = \text{angle } B$.

$\therefore \text{Angle } A = 33^\circ 34' 40''$.

8. In a plane triangle, ABC , given the side (a), opposite the angle $A = 7891.23456$; the side (b), opposite the angle $B = 12345.67891$; and the side (c), opposite the angle $C = 8912.34567$, to find the area and also the angle A .

Put $2s = a + b + c$. It is well known that the area
 $= \sqrt{s(s-a)(s-b)(s-c)}$.

$$s = 14574.62957 = 10^4 \times \downarrow 3,9,1,2,1,4,2,2, = \downarrow 958751939, \downarrow$$

$$s - a = 6683.39501 = 10^3 \times 6 \downarrow 1,1,2,6,0,7,1,5, = \downarrow 880782399, \downarrow$$

$$s - b = 2228.95066 = 10^3 \times 2 \downarrow 1,1,3,1,2,4,7,4, = \downarrow 770967363, \downarrow$$

$$s - c = 5662.28390 = 10^3 \times 5 \downarrow 1,2,9,1,8,3,0,8, = \downarrow 864201688, \downarrow$$

$$\begin{array}{r}
 2) 3474703389 \\
 \hline
 1737351695
 \end{array}$$

$$+ 1737351695 = 10^7 \times 3 \downarrow 1,6,0,9,2,3,8,3, = 35062539.6, \text{ area.}$$

$$- 1611890567 \sim 10^7$$

$$\underline{125461128}$$

$$109866750 \sim 3$$

$$\underline{15594378}$$

$$9531497 \sim (1,$$

$$\underline{6062881}$$

$$5970498 \sim (6,$$

$$\underline{9,2,3,8,3,}$$

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$b = 12345.67891 = 10^4 \times \downarrow 2,2,0,2,0,0,0,2, = \downarrow \overline{942153486,}$$

$$c = 8912.34567 = 10^3 \times 8 \downarrow 1,1,2,7,3,6,4,5, = \downarrow \overline{909564982,}$$

$$1851718468$$

$$s - b = 770967363$$

$$s - c = 864201688$$

$$+ 1635169051$$

$$bc = - 1851718468$$

$$2) - \underline{216549417}$$

$$- 108274709 = \downarrow z, \text{ to be referred to presently.}$$

$$10 \sim + \underline{230270081}$$

$$+ 121995372 = 3 \downarrow 1,2,6,0,7,2,2,9, = 3.38682336$$

$$3 \sim \underline{109866750}$$

$$12128622$$

$$\underline{9531497} (1,$$

$$2597125$$

$$\underline{1990166} (2,$$

$$606959$$

$$\underline{599730} (6,$$

$$\underline{0,7,2,2,9,}$$

$$\therefore 338682336 = \sin \frac{A}{2}.$$

If x be the length of a circular arc, and z its sine, to radius 1 (it may be repeated here), then

$$x = z + \frac{1}{2.3} z^3 + \frac{1.3}{2.4.5} z^5 + \frac{1.3.5}{2.4.6.7} z^7 + \frac{1.3.5.7}{2.4.6.8.9} z^9 + \dots$$

By adding the corresponding values of 1, 3, 5, 7, reduced to the eight position, and subtracting the values of 2, 4, 6, reduced to the same position, the following arrangement is readily formed, and may be applied in similar cases.

$$\frac{1}{2.3} = - 179184951$$

$$\frac{1.3}{2.4.5} = - 259039733$$

$$\frac{1.3.5}{2.4.6.7} = - 310921720$$

$$\frac{1.3.5.7}{2.4.6.8.9} = - 349408234$$

$$\frac{1.3.5.7.9}{2.4.6.8.10.11} = - 380012893$$

$$\downarrow \overline{z} = - \begin{array}{r} 108274709 \\ 3 \end{array}$$

$$\frac{1}{2.3} = - \begin{array}{r} 324824127 \\ 179184951 \end{array}$$

$$- 504009078 = \frac{6}{10^3} \downarrow 0,7,6,5,0,9,0,3, = .00647480.$$

$$\downarrow \overline{z} = - \begin{array}{r} 108274709 \\ 5 \end{array}$$

$$\frac{1.3}{2.4.5} = - \begin{array}{r} 541373545 \\ 259039733 \end{array}$$

$$- 800413278 = \frac{3}{10^4} \downarrow 1,1,2,7,3,8,0,6, = .00033421.$$

$$\downarrow \overline{z} = - \begin{array}{r} 108274709 \\ 7 \end{array}$$

$$\frac{1.3.5}{2.4.6.7} = - \begin{array}{r} 757922963 \\ 310921720 \end{array}$$

$$- 1068844683 = \frac{2}{10^5} \downarrow 1,3,6,7,1,0,4,5, = .00002282.$$

$$\begin{array}{r} \downarrow z, \mid = - \quad 108274709 \\ \quad \quad \quad 9 \\ \hline 1.3.5.7 \quad - \quad 974472381 \\ 2.4.6.8.9. \quad - \quad 349408234 \\ \hline \quad - \quad 1323880615 = \frac{1}{10^6} \downarrow 6,0,5,5,1,1,1,4. = .000000178. \end{array}$$

$$\begin{array}{r} \downarrow z, \mid = - \quad 108274709 \\ \quad \quad \quad 11 \\ \hline 1.3.5.7.9 \quad - \quad 1191021799 \\ 2.4.6.8.10.11 \quad - \quad 380012893 \\ \hline \quad - \quad 1571034692 = \frac{1}{10^7} \downarrow 4,2,7,4,0,0,3,6, = .000000015. \end{array}$$

$$\text{Approximate value of next step} = \frac{15 \times 11 \times 11 \times (.33)^2}{12 \cdot 13} = 1.$$

$$\begin{array}{r} \text{Sine} = .33868234 \\ \quad \quad 647480 \\ \quad \quad 33421 \\ \quad \quad 2282 \\ \quad \quad 178 \\ \quad \quad 15 \\ \quad \quad 1 \\ \hline \end{array}$$

$$\text{Arc} = .34551611 \text{ of } 19^\circ 47' 47''.8122$$

$$\therefore \text{Angle } A = 39^\circ 35' 35''.6244$$

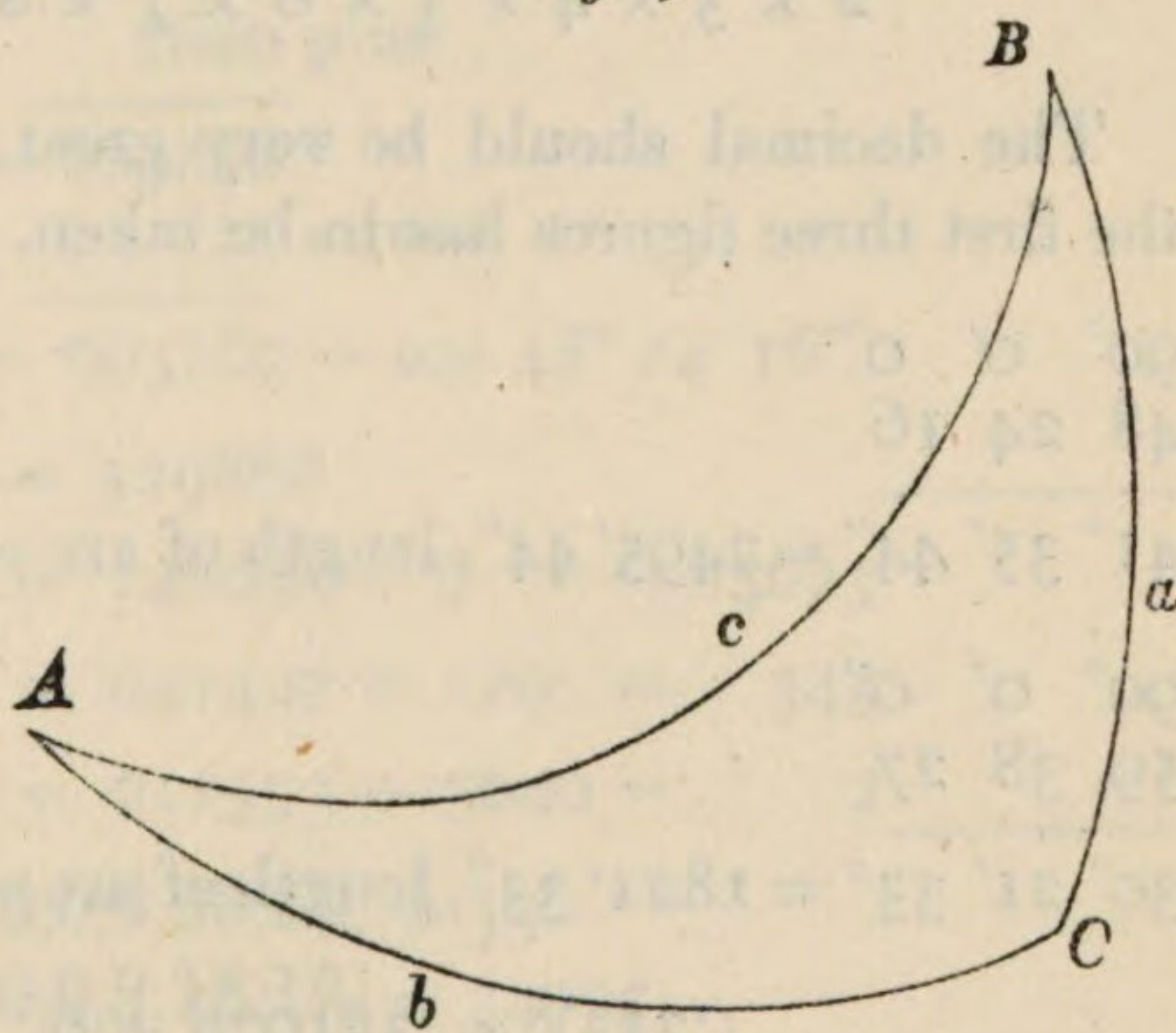
PART VI.

DUAL ARITHMETIC APPLIED TO THE SOLUTION
OF THE DIFFERENT CASES OF SPHERICAL
TRIANGLES.

RIGHT-ANGLED SPHERICAL TRIANGLES.

Fig. 9.

The angles of the spherical triangle may be represented by the letters at their vertices A , B , C ; and the sides opposite to them by the small letters a , b , c .



In any right-angled triangle ABC ,

$$\sin a = \sin c \sin A,$$

$$\sin b = \sin c \sin B,$$

$$\cos c = \cos a \cos b,$$

$$\cos A = \cos a \sin B,$$

$$\cos B = \cos b \sin A.$$

1. In the right-angled spherical triangle ABC are given,
 $a = 48^\circ 24' 16''$; $b = 59^\circ 38' 27''$; to find the hypotenuse c .

It is a question for the operator to decide, whether the sine or cosine of an arc ought to be found at once, or by employing two operations; if a table of the squares and cubes of numbers be convenient, perhaps one operation is more easily managed. It is easily observed, when the three first figures of the length of an arc are raised to a power sufficiently high, from the following continued products,

$$\begin{aligned} 2 &= 2 \\ 2 \times 3 &= 6 \\ 2 \times 3 \times 4 &= 24 \\ 2 \times 3 \times 4 \times 5 &= 120 \\ 2 \times 3 \times 4 \times 5 \times 6 &= 720 \\ 2 \times 3 \times 4 \times 5 \times 6 \times 7 &= 5040 \\ 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 &= 40320 \\ 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 &= 362880. \end{aligned}$$

The decimal should be very great, when the ninth power of the first three figures has to be taken.

$$\begin{array}{r} 90^\circ \quad 0' \quad 0'' \\ 48 \quad 24 \quad 16 \\ \hline \end{array}$$

$$41^\circ 35' 44'' = 2495' 44'', \text{ length of arc} = .725979 = .725 \downarrow 0,0,1,3,5,$$

$$\begin{array}{r} 90^\circ \quad 0' \quad 0'' \\ 59 \quad 38 \quad 27 \\ \hline \end{array}$$

$$30^\circ 21' 33'' = 1821' 33'', \text{ length of arc} = .529868 = .529 \downarrow 0,0,1,6,4,$$

$$(.725)^3 = .381078 \div 6 = 63513$$

$$(.725)^5 = .200304 \div 120 = 1669$$

$$(.725)^7 = .105285 \div 5040 = 21$$

$$\begin{array}{cccccc} 2) & 3) & 4) & 5) & 6) & 7) \\ 105285 & 52643 & 17548 & 4387 & 877 & 149 & 21 \end{array}$$

The nearest whole number to the quotient of 105285 divided by $2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040 = 21$.

2) 200304	3) 100152	4) 33384	5) 834 6 8... ~ 1 8385 3.... ~ 3	↓ 0,0,5,15,25, 166 9.. 8.. 1677 3 1680
--------------	--------------	-------------	---	---

2) 381078	3) 1905 39 191. ~ 1 ~ 19111 2 57... ~ 3 ~ 191283 10..... ~ 5 ~	↓ 0,0,3,9,15, 635 13.. 191.. 6370 4... 57... 63761 10 63771
--------------	--	--

$$\begin{array}{r}
 \text{Length of arc} = \cdot 725979 \\
 \quad \quad \quad 63771 \text{ minus} \\
 \hline
 \cdot 662208 \\
 \quad \quad \quad 1680 \text{ plus} \\
 \hline
 \cdot 663888 \\
 \quad \quad \quad 21 \text{ minus} \\
 \hline
 \end{array}$$

$$\sin 41^\circ 35' 44'' = \cdot 663867 = \cos 48^\circ 24' 16''.$$

$$\text{Again, } \cdot 529 \downarrow 0,0,1,6,4, = \cdot 529868$$

$$(\cdot 529)^3 = \cdot 148036 \div 6 = 24673$$

$$(\cdot 529)^5 = \cdot 041427 \div 120 = 345$$

$$(\cdot 529)^7 = \cdot 011593 \div 5040 = 2$$

$$345 \downarrow 0,0,5,30,20, = 348$$

$$24673 \downarrow 0,0,3,18,12, = 24795.$$

$$\begin{array}{r}
 \text{Length of arc} = \cdot 529868 \\
 \quad \quad \quad 24795 \text{ minus} \\
 \hline
 \cdot 505073 \\
 \quad \quad \quad 348 \text{ plus} \\
 \hline
 \cdot 505421 \\
 \quad \quad \quad 2 \text{ minus} \\
 \hline
 \end{array}$$

$$\cos 59^\circ 38' 27'' = \cdot 505419 = \sin 30^\circ 21' 33''.$$

$\therefore .663867 \times .505419 = .335532 = .335 \downarrow 0,0,1,5,9$, = cosine of c , the required side; from which the degrees, minutes, &c. in c has to be found.

$$(.335)^3 = .037595$$

$$(.335)^5 = .004219$$

$$(.335)^7 = .000473$$

$$\frac{1}{2} \\ 473$$

$$\frac{3}{4} \\ 237$$

$$\frac{5}{6} \\ 178$$

$$7 \\ 148$$

$$21$$

$$\frac{1}{2} \\ 4219$$

$$\frac{3}{4} \\ 2109$$

$$\begin{array}{r} 5 \\ 1582 \end{array} \Big| \sim 1 \sim$$

$$\begin{array}{r} 1590. \\ 1 \dots \end{array} \Big| \sim 5 \sim$$

$$\downarrow 0,0,5,25,45,$$

$$\begin{array}{r} 316 \end{array} \Big| \dots$$

$$\begin{array}{r} 318. \\ 1 \end{array} \Big| \dots$$

$$319$$

$$\frac{1}{2} \\ 37595$$

$$\begin{array}{r} 3 \\ 1879 \end{array} \Big| 8$$

$$19 \dots \Big| \sim 1 \sim$$

$$\begin{array}{r} 18855 \\ 9 \dots \end{array} \Big| \sim 5 \sim$$

$$\begin{array}{r} 18882. \\ 2 \dots \end{array} \Big| \sim 9 \sim$$

$$\downarrow 0,0,3,15,27,$$

$$\begin{array}{r} 626 \end{array} \Big| 6 \dots$$

$$\begin{array}{r} 6285 \\ 9 \end{array} \Big| \dots$$

$$\begin{array}{r} 6294. \\ 2 \end{array} \Big| \dots$$

$$6296$$

$$\text{Length of sine} = .335532$$

$$6296$$

$$319$$

$$21$$

$$\text{Length of arc} = .342168 = 1176' 18'' = 19^\circ 36' 18''$$

$$\therefore c = 70^\circ 23' 42''.$$

2. In a spherical triangle ABC , right-angled at C , are given $b = 46^\circ 18' 23''$, $A = 34^\circ 27' 39''$, to find the other angle B .

$$\cos B = \cos b \sin A.$$

$$\begin{array}{r} 90^{\circ} \quad 0' \quad 0'' \\ 46 \quad 18 \quad 23 \\ \hline \end{array}$$

$$43^{\circ} 41' 37'' = 2621' 37'', \text{ length of arc} \\ = \cdot 762597 = 762 \downarrow 0,0,0,7,8,4,$$

$$(\cdot 762)^3 = \cdot 442451 \div 6 = 73742; \downarrow 0,0,0,21,24,12, = 73916$$

$$(\cdot 762)^5 = \cdot 256907 \div 120 = 2141; \downarrow 0,0,0,35,40,20, = 2149$$

$$(\cdot 762)^7 = \cdot 149171 \div 5040 = 30; \downarrow 0,0,0,49,56,28, = 30$$

As a matter of form, $\downarrow 0,0,0,49,56,28$, is affixed to 30, the nearest whole number to the quotient of 149171 divided by $2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$. It is evident that 30 multiplied by $\downarrow 0,0,0,49,56,28$, will not amount to 31, which is easily shown:

$$\begin{array}{r} 7) \\ 210 \dots | \\ 14 \dots | \sim 7 \sim \end{array} \quad \begin{array}{r} \downarrow 0,0,0,49,56,28, \\ 30 \dots | \dots \\ \dots | \dots \\ \hline 30 \end{array}$$

$$\begin{array}{r} \text{Length of arc} = \cdot 762597 \\ 73916 \text{ minus} \\ \hline \cdot 688681 \\ 2149 \text{ plus} \\ \hline \cdot 690830 \\ 30 \text{ minus} \\ \hline \end{array}$$

$$\sin 43^{\circ} 41' 37'' = \cdot 690800 = \cos 46^{\circ} 18' 23''$$

$$34^{\circ} 27' 39'', \text{ arc of which} = \cdot 601454 = \cdot 600 \downarrow 0,0,2,4,2,$$

$$(\cdot 6)^3 = \cdot 216000 \div 6 = 36000, \text{ mult. by } \downarrow 0,0,6,12,6, \text{ gives } 36262.$$

$$(\cdot 6)^5 = \cdot 077760 \div 120 = 648, \text{ mult. by } \downarrow 0,0,10,20,10, \text{ gives } 655.$$

$$(\cdot 6)^7 = \cdot 027994 \div 5040 = 6, \text{ which has not to be multiplied;}$$

$$\begin{array}{r} \therefore \text{Length of arc} = \cdot 601454 \\ 36262 \text{ minus} \\ \hline \cdot 565192 \\ 655 \text{ plus} \\ \hline \cdot 565847 \\ 6 \text{ minus} \\ \hline \end{array}$$

$$\text{sine of } 34^{\circ} 27' 39'' = \cdot 565841$$

$$\therefore \cdot 565841 \times \cdot 6908 = \cdot 390883 = \cos B = 39 \downarrow 0,0,2,2,6,5,$$

$$(\cdot 39)^3 = \cdot 059319$$

$$(\cdot 39)^5 = \cdot 009022$$

$$(\cdot 39)^7 = \cdot 001372$$

$$(\cdot 39)^9 = \cdot 000209$$

$$\frac{1}{2} \\ 209$$

$$\frac{3}{4} \\ 105$$

$$\frac{5}{6} \\ 79$$

$$\frac{7}{8} \\ 66$$

$$9) \\ 58$$

$$6;$$

$$\frac{1}{2} \\ 1372$$

$$\frac{3}{4} \\ 686$$

$$\frac{5}{6} \\ 515$$

$$7) \\ 429$$

$$61;$$

$$61 \downarrow 0,0,14,14,42,35, = 62.$$

All the factors except the first may be neglected, as the employment of the other factors will not increase 62 a single unit in the sixth decimal place. This is easily shown.

$$\begin{array}{r} 7) \\ 427. \\ 8 \dots \end{array} \sim 2$$

$$\begin{array}{r} \downarrow 0,0,14, \\ 61. \dots \\ 1. \dots \\ \hline 62 \end{array}$$

$$\frac{1}{2} \\ 9022$$

$$\frac{3}{4} \\ 4511$$

$$5) \\ 3383$$

$$\downarrow 0,0,10,10,30,25, \\ 677$$

$$\therefore 677 \downarrow 0,0,10,10, = 685.$$

$$\frac{1}{2} \\ 59319$$

$$3 \\ 29660$$

$$\downarrow 0,0,6,6,18,15, \\ 9887$$

$$\therefore 9889 \downarrow 0,0,6,6,18, = 9954.$$

$$\begin{array}{r} \cdot 390883 \\ 9954 \\ 685 \\ 62 \\ 6 \\ \hline \end{array}$$

$\cdot 401590 = \text{arc of } 1380' 34'' = 23^\circ 0' 34'',$
which taken from 90° gives $69^\circ 59' 26'' = \text{the angle } B.$

3. *In an oblique spherical triangle the three sides are*

$$a = 68^{\circ} 46' 2''$$

$$b = 43 37 38$$

$$c = 37 10 0$$

required the angle A.

It is well known that, if s be half the sum of the three sides, then :

$$\sin \frac{A}{2} = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}}$$

$$\cos \frac{A}{2} = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}}$$

In this example, $s = 74^{\circ} 46' 50''$ $\sin = \cdot 964938$.

$$\sin a = \cdot 932117, \sin b = \cdot 689964, \sin c = \cdot 604136.$$

$$s - b = 31^{\circ} 9' 12'' \sin = \cdot 517330;$$

$$s - c = 37 36 50 \sin = \cdot 610337;$$

$$s - a = 6 0 48 \sin = \cdot 104760.$$

$$\therefore \cos \frac{A}{2} = \sqrt{\frac{\cdot 964938 \times \cdot 104760}{689964 \times \cdot 604136}}$$

The object of working out this example, is to dispense altogether with the common and laborious operations of multiplication, division, and of the square root. From a table of the squares of numbers, or otherwise, it is readily found that

$$\cdot 964938 \text{ stands between } (\cdot 982)^2 \text{ and } (\cdot 983)^2,$$

$$\cdot 104760 \quad \quad \quad \text{,,} \quad (\cdot 323)^2 \text{ and } (\cdot 324)^2,$$

$$\cdot 689964 \quad \quad \quad \text{,,} \quad (\cdot 830)^2 \text{ and } (\cdot 831)^2,$$

$$\cdot 604136 \quad \quad \quad \text{,,} \quad (\cdot 777)^2 \text{ and } (\cdot 778)^2.$$

It is also evident that

$$\cdot 77 \text{ is contained in } \cdot 924 \text{ exactly } 1\cdot 2 \text{ times}$$

$$\cdot 805 \quad \quad \quad \text{,,} \quad \cdot 322 \quad \quad \quad \text{,,} \quad \cdot 4 \quad \quad \quad \text{,,}$$

Now let P, Q, R, S , be factors with the following properties, namely :—

$$(\cdot 924)^2 \downarrow P^2 = \cdot 964938$$

$$(\cdot 322)^2 \downarrow Q^2 = \cdot 104760$$

$$(\cdot 77)^2 \downarrow R^2 = \cdot 604136$$

$$(\cdot 805)^2 \downarrow S^2 = \cdot 689964$$

$$\therefore \cos \frac{A}{2} = \frac{\cdot 924 \downarrow P \times \cdot 322 \downarrow Q}{\cdot 77 \downarrow R \times \cdot 805 \downarrow S} = \frac{\cdot 48 \downarrow P \downarrow Q}{\downarrow R \downarrow S}$$

When the factors P, Q, R, S , are found, $\cos \frac{A}{2}$ becomes known without extracting the square root, and without multiplying and dividing the given cosines.

The work necessary to find P may be arranged as follows :—

$\begin{array}{r} 2 \\ 17075 \overline{) 52} \\ 102453 \quad \sim 6 \sim \end{array}$	$\begin{array}{r} 964938 = (\cdot 924)^2 \downarrow 0, 12, 2, 8, 18, 12, \\ 85 \overline{) 3776} = (\cdot 924)^2 \\ 10 \overline{) 2453} = 12 \\ \quad 56 \overline{) 35} = \frac{11}{2} \\ \quad \quad 1 \overline{) 88} = \frac{10}{2} \\ \quad \quad \quad 4 = \frac{3}{4} \end{array}$
$\begin{array}{r} 1924 \overline{) 112} \\ 1924 \quad \sim 1 \sim \end{array}$	$\begin{array}{r} 962 \overline{) 056} \\ 1 \overline{) 924} = 2 \\ \quad 1 = \frac{1}{2} \end{array}$
$\begin{array}{r} 19279 \overline{) 62} \\ 771 \dots \quad \sim 4 \sim \end{array}$	$\begin{array}{r} 9639 \overline{) 81} \dots \\ 7 \overline{) 71} \dots \end{array}$
$\begin{array}{r} 192950 \overline{) 4} \\ 174 \dots \quad \sim 9 \sim \end{array}$	$\begin{array}{r} 96475 \overline{) 2} \dots \\ 17 \overline{) 4} \dots \end{array}$
$\begin{array}{r} 1929852 \overline{) } \\ 12 \dots \quad \sim 6 \sim \end{array}$	$\begin{array}{r} 964926 \overline{) } \dots \\ 12 \overline{) } \dots \end{array}$
	$964938, \text{ the number required.}$

$$\therefore P = \downarrow 0, 6, 1, 4, 9, 6, +$$

$$Q = \downarrow 0, 0, 5, 1, 6, 0, + \text{ found in the same way.}$$

$$R = \downarrow 0, 0, 9, 3, 9, 0, -$$

$$S = \downarrow 0, 3, 1, 5, 0, 4, -$$

$$\downarrow 0, 3, \bar{4}, \bar{3}, 6, 2, = \frac{\downarrow P \downarrow Q}{\downarrow R \downarrow S}$$

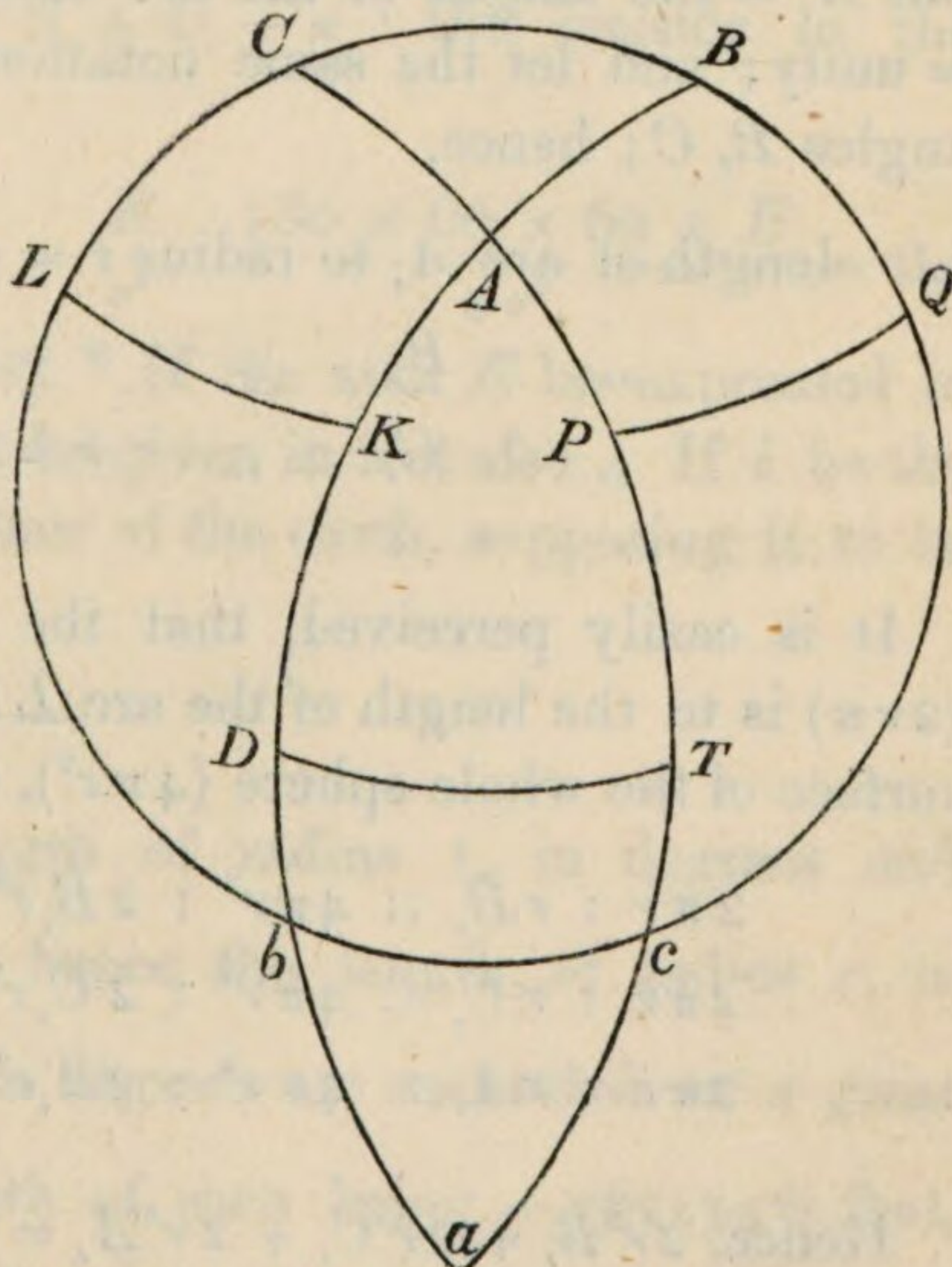
$$\cdot 48 \downarrow 0, 3, \bar{4}, \bar{3}, 6, 2, = \cdot 492454 = \cos \frac{A}{2};$$

$$\therefore \frac{A}{2} = 60^\circ 29' 53''.$$

THE AREA OF SPHERICAL TRIANGLES AND THE SPHERICAL EXCESS.

Let $BQcbLC$ represent the earth, a sphere, radius $= r$, and ABC a spherical triangle, formed by the three great circles BAb , CAc , CBc ; and let $BL, BK; CQ, CP; AD, AT$, quadrants. The spherical angle A may be measured by the arc DT , the spherical angle B by the arc LK , and the spherical angle C by the arc PQ , being all arcs of great circles. The spherical triangle abc , part of the lune Aa , is supposed to be unwrapped from the obverse hemisphere.

Fig. 10.



The triangle abc is equal to the triangle ABC , because the spherical angle A is equal to the spherical angle a , and

$$Ab + AB = \text{semicircle} = Ab + bc,$$

$$Ac + AC = \text{semicircle} = Ac + ac,$$

$$\therefore AB = ab$$

$$AC = ac.$$

Then it is evident that the area of the lune $AbacA$, that is, the lune Aa , is equal to the area of ABC + area of Abc ; and the sum of the areas of the lunes Aa , Bb , Cc , is equal to the area of the hemisphere $BcbC$. On these considerations the following calculations depend. It is well known and easily proved, that the surface of a sphere is equal to four times the area of any one of its great circles;

$$\therefore 2\pi r^2 = \text{area of the hemisphere } BCLb'cQ.$$

Let A° represent the degrees, &c. in the spherical angle A , and put A_1 = the length of the arc that measures this angle, radius = unity; and let the same notation be applied to the spherical angles B , C ; hence,

$$\left. \begin{array}{lll} \text{length of arc } A_1 \text{ to radius } r = rA_1 = DT, \\ \text{,, } B_1 \text{ ,, } r = rB_1 = LK, \\ \text{,, } C_1 \text{ ,, } r = rC_1 = PQ, \end{array} \right\} (\text{see fig. 10.})$$

It is easily perceived, that the circumference of the sphere ($2\pi r$) is to the length of the arc $LK = rB_1$; so is the area of the surface of the whole sphere ($4\pi r^2$), to the area of the lune Bb , or

$$2\pi r : rB_1 :: 4\pi r^2 : 2B_1r^2 = \text{area of } BLbKB;$$

$$2\pi r : rC_1 :: 4\pi r^2 : 2C_1r^2 = \text{area of } CQcPC;$$

$$2\pi r : rA_1 :: 4\pi r^2 : 2A_1r^2 = \text{area of } ADaTA.$$

Hence, $2r^2B_1 + 2r^2C_1 + 2r^2A_1$ = half the surface of the sphere + twice the area of the triangle ABC , fig. 10.

Putting E for the area of the spherical triangle, then,

$$2\pi r^2 + 2E = 2r^2 B_1 + 2r^2 C_1 + 2r^2 A_1 ;$$

$$\therefore E = (A_1 + B_1 + C_1 - \pi) r^2, \quad (1.)$$

Let E be represented by e , when $r = 1$, and suppose the angles to be measured in degrees, then (1) becomes

$$e = (A^\circ + B^\circ + C^\circ - 180^\circ), \quad (2.)$$

(2) is Girard's Theorem, and shows that the area of a spherical triangle may be represented by e , the excess of the sum of its three angles above two right angles. e is technically termed the *Spherical Excess*.

$$\text{From (1), } A_1 + B_1 + C_1 - \pi = \frac{E}{r^2}, \quad (3.)$$

(3) expresses the spherical excess, in the lengths of arcs to radius unity; A, B, C , being the lengths of the arcs that measure the angles $A B C$ to radius unity.

$\pi : 180 \times 60 \times 60 :: A_1 + B_1 + C_1 - \pi : \text{the seconds in the spherical excess}$

$$\therefore \pi : 180 \times 60 \times 60 :: \frac{E}{r^2} : \frac{180 \times 60 \times 60 \times E}{\pi r^2} = \text{the}$$

seconds in the spherical excess. If the area E be expressed in square feet, the radius r must be given in feet also. If l be the length of a degree on the surface of the earth, supposing it to be

a sphere, then $r = \frac{180l}{\pi}$ for,

$\pi : 180^\circ :: 1 : \frac{180}{\pi} = \text{length of radius 1, in degrees and decimal parts of a degree; hence the length of radius } r, \text{ in degrees,} = \frac{180}{\pi} \text{ also, but these degrees are measured on a great circle of the earth, the length of each being} = 365154.6 \text{ feet, which put} = l.$

$$\therefore r = \frac{180l}{\pi}, \text{ and } r^2 = \frac{180^2 l^2}{\pi^2}.$$

$\therefore$ The spherical excess in seconds

$$= \frac{180 \times 60 \times 60 \times E}{\pi r^2} = \frac{60 \times 60 \times \pi E}{180 l^2}.$$

$$\therefore \text{The spherical excess in seconds, } = \frac{20\pi E}{l^2}, (4.)$$

If (4) be put in a logarithmic form it gives Dalby's rule.

E , the area of the spherical triangle ABC , being in square feet.

4. Given $E = 124797955000$, square feet, the area of the spherical triangle ABC , fig. 10, to find the spherical excess.

$$l = 365154.6 = 10^5 \times 3 \downarrow 2,0,5,9,2,0,5,5,$$

$$E = 12479795500 = 10^{11} \downarrow 2,3,1,0,5,5,0,3,$$

$$\pi = 3 \downarrow 0,4,6,3,1,9,3,0,$$

(4), may be written

$$\frac{10^{11} \times 20 \times 3 \downarrow \pi', \downarrow E'}{10^{10} \times 9 \downarrow 2l'} = \frac{2}{3} \frac{10^2 \downarrow \pi', \downarrow E'}{\downarrow 2l'}$$

$$E' = \downarrow 2,3,1,0,5,5,0,3, = 22153701 \text{ reduced to the eight position.}$$

$$\pi' = \downarrow 0,4,6,3,1,9,3,0, = 4611992 \quad \text{,,} \quad \text{,,} \quad \text{,,}$$

$$10^2 = 460540162 \quad \text{,,} \quad \text{,,} \quad \text{,,}$$

$$l' = \downarrow 2,0,5,9,2,0,5,5, \} 487305855$$

$$= \downarrow 19654824, \} ; 2l' \} = 39309648$$

$$447996207 = 10 \times 8 \downarrow 1,0,2,4,0,1,1,5,$$

$$230270081 \sim 10$$

$$217726126$$

$$207954604 \sim 8$$

$$9771522$$

$$9531497 \sim \downarrow 1,$$

$$240025$$

$$199910 \sim (0,2,$$

$$4,0,1,1,5$$

$$\downarrow 1, 0, 2, 4, 0, 1, 1, 5, = 1^{\circ} 10' 26.4332''$$

$$\begin{array}{r} 80 \\ \hline 88.21146560 \end{array}$$

2

$$3 \) \ 176.42293120$$

$$\begin{array}{r} \hline 58.80764373 \end{array}$$

$\therefore$ The spherical excess = $58''.80764373$.

The same result may be found in the following manner, by continued reductions:—

$$E' = \downarrow 2, \quad 3, \quad 1, \quad 0, \quad 5, \quad 5, \quad 0, \quad 3,$$

$$\pi' = \downarrow 0, \quad 4, \quad 6, \quad 3, \quad 1, \quad 9, \quad 3, \quad 0,$$

$$\begin{array}{r} \hline \downarrow 2, \quad 7, \quad 7, \quad 3, \quad 6, \quad 14, \quad 3, \quad 3, \end{array}$$

$$2\ell' = \downarrow 4, \quad 0, \quad 10, \quad 18, \quad 4, \quad 0, \quad 10, \quad 10,$$

$$\begin{array}{r} \hline \downarrow \bar{2}, \quad 7, \quad \bar{3}, \quad \bar{15}, \quad 2, \quad 14, \quad \bar{7}, \quad \bar{7}, \end{array} \quad (5.)$$

$$= \downarrow \bar{2}, \quad 0, \quad 67, \quad \bar{15}, \quad \bar{26}, \quad \bar{14}, \quad \bar{49}, \quad \bar{56},$$

$$= \downarrow \bar{2}, \quad 0, \quad 0, \quad 655, \quad \bar{26}, \quad \bar{14}, \quad \bar{317}, \quad \bar{391},$$

$\therefore$ (5) is reduced to $\downarrow \bar{2}, 6, 5, 4, 8, 7, 6, 6,$

$$6550$$

$$\begin{array}{r} 26 \\ \hline \end{array}$$

$$65240$$

$$\begin{array}{r} 14 \\ \hline \end{array}$$

$$652260$$

$$\begin{array}{r} 317 \\ \hline \end{array}$$

$$6519430$$

$$\begin{array}{r} 391 \\ \hline \end{array}$$

$$6519039 = \downarrow 0, 6, 5, 4, 8, 7, 6, 6,$$

R

WORK UNABRIDGED.

$$\begin{array}{r}
 1 \overline{00000000} \overline{00} \overline{00} \text{ minus} \\
 - \overline{20406080} \overline{10} \overline{00} \\
 \hline
 79593919 \overline{00} \overline{00} \\
 3050709 \overline{00} \overline{11} \text{ plus}
 \end{array}$$

$$\downarrow \bar{2}, = \begin{array}{r}
 82 \overline{64} \overline{46} \overline{28} \overline{11} \sim 1 \\
 495 \overline{86} \overline{77} \overline{7} \sim 6 \\
 \quad \overline{12} \overline{39} \overline{66} \overline{9} \sim 15 \\
 \quad \quad \overline{16} \overline{52} \overline{9} \sim 20 \\
 \quad \quad \quad \overline{124} \sim 15
 \end{array}$$

$$\begin{array}{r}
 877 \overline{289} \overline{380} \sim 1 \\
 438 \overline{644} \overline{7} \sim 5 \\
 \quad 8 \overline{773} \sim 10 \\
 \quad \quad 9 \sim 10
 \end{array}$$

$$\begin{array}{r}
 8816 \overline{8460} \overline{9} \dots \sim 1 \\
 \quad 3 \overline{5267} \overline{4} \dots \sim 4 \\
 \quad \quad 5 \overline{3} \dots \sim 6
 \end{array}$$

$$\begin{array}{r}
 88203 \overline{7336} \\
 70 \overline{563} \sim 8 \\
 61 \overline{74} \sim 7 \\
 \quad 5 \overline{29} \sim 6 \\
 \quad \quad 5 \overline{3} \sim 6
 \end{array}$$

$$882114655$$

$$\begin{array}{r}
 8.82114655 \\
 \quad \quad \quad 200
 \end{array}$$

$$\begin{array}{r}
 3 \overline{) 176.422931000} \\
 \underline{58.8076437}
 \end{array}$$

The spherical excess the same as before found.

IMPORTANT MISCELLANEOUS PROBLEMS,

SOLVED BY

DUAL ARITHMETIC.

1. Find the hyperbolic logarithm of 1.95 to seven places of decimals.

$$\begin{aligned} \epsilon &= \downarrow 10, & 4, & 7, & 1, & 0, & 0, & 3, & 8, \\ &= \downarrow 0, & 104, & \overline{33}, & \overline{9}, & \overline{90}, & \overline{50}, & \overline{7}, & \overline{22}, \\ &= \downarrow 0, & 0, & 1007, & \overline{9}, & \overline{506}, & \overline{466}, & \overline{631}, & \overline{750}, \\ &= \downarrow 0, & 0, & 0, & 10061, & \overline{506}, & \overline{466}, & \overline{4659}, & \overline{5785}, \end{aligned}$$

$$\begin{array}{r} 100610 \\ \quad 506 \\ \hline 1001040 \\ \quad 466 \\ \hline 10005740 \\ \quad 4659 \\ \hline 100010810 \\ \quad 5785 \\ \hline 100005025 \end{array}$$

This reduction is made by employing the property :

$$\begin{aligned} \downarrow 1, &= \downarrow 0, & 10, & \overline{4}, & \overline{1}, & \overline{9}, & \overline{5}, & \overline{1}, & \overline{3}, \\ \downarrow 0,1, &= \downarrow 0, & 0, & 10, & 0, & \overline{4}, & \overline{4}, & \overline{6}, & \overline{7}, \\ \downarrow 0,0,1, &= \downarrow 0, & 0, & 0, & 10, & 0, & 0, & \overline{4}, & \overline{5}, \\ \downarrow 0,0,0,1, &= \downarrow 0, & 0, & 0, & 0, & 10, & 0, & 0, & 0, \\ \&c. &= & & & & & & & \&c. \end{aligned}$$

$$\begin{array}{r}
 1.95 = \downarrow 7, \quad 0, \quad 0, \quad 6, \quad 5, \quad 8, \quad 1, \quad 4, \\
 \downarrow 0, \quad 70, \quad \overline{28}, \quad \overline{1}, \quad \overline{58}, \quad \overline{27}, \quad \overline{6}, \quad \overline{17}, \\
 \downarrow 0, \quad 0, \quad 672, \quad \overline{1}, \quad \overline{338}, \quad \overline{307}, \quad \overline{426}, \quad \overline{507}, \\
 \downarrow 0, \quad 0, \quad 0, \quad 6719, \quad \overline{338}, \quad \overline{307}, \quad \overline{3114}, \quad \overline{3867}.
 \end{array}$$

$$\begin{array}{r}
 67190 \\
 \underline{338} \\
 668520 \\
 \underline{307} \\
 6682130 \\
 \underline{3114} \\
 66790160 \\
 \underline{3867} \\
 66786293
 \end{array}$$

If 66786293 be divided by 100005025, the quotient is .6678293, the hyperbolic log. of 1.95, as required.

2. *Required the hyperbolic logarithm of 1.95 to five places of decimals.*

$$\begin{array}{r}
 1.95 = \downarrow 7, \quad 0, \quad 0, \quad 6, \quad 5, \quad 8, \\
 = \downarrow 0, \quad 70, \quad \overline{28}, \quad \overline{1}, \quad \overline{58}, \quad \overline{27}, \\
 = \downarrow 0, \quad 0, \quad 672, \quad \overline{1}, \quad \overline{338}, \quad \overline{307},
 \end{array}$$

$$\begin{array}{r}
 6720 \\
 \underline{1} \\
 67190 \\
 \underline{338} \\
 668520 \\
 \underline{307} \\
 668213
 \end{array}$$

$$\begin{array}{r}
 \epsilon = \downarrow 10, \quad 4, \quad 7, \quad 1, \quad 0, \quad 0, \\
 = \downarrow 0, \quad 104, \quad \overline{33}, \quad \overline{9}, \quad \overline{90}, \quad \overline{50}, \\
 = \downarrow 0, \quad 0, \quad 1007, \quad \overline{9}, \quad \overline{506}, \quad \overline{466}, \\
 = \downarrow 0, \quad 0, \quad 0, \quad 10061, \quad \overline{506}, \quad \overline{466},
 \end{array}$$

$\therefore \epsilon$ in the sixth position = 1000574.

Then 668213 divided by 1000574 gives .66783, the hyperbolic logarithm of 1.95, true to five places of decimals as required.

3. *Required the hyperbolic logarithm of 2 to seven places of decimals.*

$$\begin{array}{r}
 2 = \downarrow 7, \quad 2, \quad 6, \quad 0, \quad 7, \quad 8, \quad 2, \quad 8, \\
 = \downarrow 0, \quad 72, \quad \overline{22}, \quad \overline{7}, \quad \overline{56}, \quad \overline{27}, \quad \overline{5}, \quad \overline{13}, \\
 = \downarrow 0, \quad 0, \quad 698, \quad \overline{7}, \quad \overline{344}, \quad \overline{315}, \quad \overline{437}, \quad \overline{517}, \\
 = \downarrow 0, \quad 0, \quad 0, \quad 6973, \quad \overline{344}, \quad \overline{315}, \quad \overline{3229}, \quad \overline{4007},
 \end{array}$$

$$\begin{array}{r}
 69730 \\
 344 \\
 \hline
 693860 \\
 315 \\
 \hline
 6935450 \\
 3229 \\
 \hline
 69322210 \\
 4007 \\
 \hline
 69318203
 \end{array}$$

69318203 divided by 100005025 gives .6931472 the hyperbolic logarithm of 2 to seven places of decimals.

Because $100005025 = \downarrow 0,0,0,0,5,0,2,5$, the quotient may be found by a simple subtraction, without the use of common division.

$$\begin{array}{r}
 69318 \overline{)203} \dots \\
 \text{half} \quad 3 \overline{)46} \overline{)6} \dots \sim 5 \\
 \text{half} \quad 1 \overline{)7} \dots \sim 25 \\
 \hline
 3483
 \end{array}$$

$$\text{Hyp. log. } 2 = .69314720$$

It must be observed, that this rule only applies, when reductions are carried to the eight position.

4. *Required the hyperbolic logarithm of 10.*

Since $e = 2.718281828459 \dots$ and it was before shown that the value of x may be found in a direct way, from the equation

$$e^x = 10;$$

then x is the hyperbolic logarithm of 10; $2x$ will be the hyperbolic logarithm of 100; $3x$ the hyperbolic logarithm of 1000,

and so on. The square roots, fourth roots, eight roots, &c. of ϵ and 10, may be found by common arithmetic; the bases will then be reduced nearer and nearer to 1, and the operation simplified.

$$\begin{aligned} \text{Let} \quad & v^8 = \epsilon, \text{ and } a^8 = 10, \\ \text{then} \quad & v^{8x} = a^8; \\ & \therefore v^x = a. \end{aligned}$$

$$\begin{aligned} \text{But } v &= \downarrow 1, \quad 2, \quad 9, \quad 7, \quad 9, \quad 3, \quad 6, \quad 9, \\ &= \downarrow 0, \quad 12, \quad 5, \quad 6, \quad 0, \quad \bar{2}, \quad 5, \quad 6, \\ &= \downarrow 0, \quad 0, \quad 125, \quad 6, \quad \bar{48}, \quad \bar{50}, \quad \bar{67}, \quad \bar{78}, \\ &= \downarrow 0, \quad 0, \quad 0, \quad 1256, \quad \bar{48}, \quad \bar{50}, \quad \bar{567}, \quad \bar{703}, \end{aligned}$$

$$\begin{array}{r} 12560 \\ 48 \\ \hline 125120 \\ 50 \\ \hline 1250700 \\ 567 \\ \hline 12501330 \\ 703 \\ \hline 12500627 \end{array}$$

$$\begin{aligned} a &= \downarrow 3, \quad 0, \quad 1, \quad 8, \quad 9, \quad 3, \quad 1, \quad 4, \\ &= \downarrow 0, \quad 30, \quad \bar{11}, \quad 5, \quad \bar{18}, \quad \bar{12}, \quad \bar{2}, \quad \bar{5}, \\ &= \downarrow 0, \quad 0, \quad 289, \quad 5, \quad \bar{138}, \quad \bar{132}, \quad \bar{182}, \quad \bar{215}, \\ &= \downarrow 0, \quad 0, \quad 0, \quad 2895, \quad \bar{138}, \quad \bar{132}, \quad \bar{1338}, \quad \bar{1660}, \end{aligned}$$

$$\begin{array}{r} 28950 \\ 138 \\ \hline 288120 \\ 132 \\ \hline 2879880 \\ 1338 \\ \hline 28785420 \\ 1660 \\ \hline 28783760 \end{array}$$

Then 28783760 divided by 12500627 gives 2.302585093, the hyperbolic logarithm of 10.

$$\therefore 2x = \text{hyp. log. } 100 = 4.6051702$$

$$3x = \text{,, } \text{,, } 1000 = 6.9077553$$

$$4x = \text{,, } \text{,, } 10000 = 9.2103404, \&c. \&c.$$

5. *Required the hyperbolic log. of 3, 4, 5, the log. of 2 being given = .693147, to five places of decimals.*

$$\begin{array}{r} 2 \) \ 3.000000 \\ \hline 1.500000 \end{array}$$

$$= \downarrow 4, 2, 4, 3, 3, 2,$$

$$= \downarrow 0, 42, \overline{12}, \overline{1}, \overline{33}, \overline{18},$$

$$= \downarrow 0, 0, 408, \overline{1}, \overline{201}, \overline{186},$$

$$\epsilon = \downarrow 10, 4, 7, 1, 0, 0,$$

$$= \downarrow 0, 104, \overline{33}, \overline{9}, \overline{90}, \overline{50},$$

$$= \downarrow 0, 0, 1007, \overline{9}, \overline{506}, \overline{466},$$

$$= 1000574, \text{ a result before found.}$$

$$\begin{array}{r} 4080 \\ \text{I} \\ \hline 40790 \\ 201 \\ \hline 405890 \\ 186 \\ \hline 405704 \end{array}$$

405704 divided by 1000574 gives .40547 = log. of 1.5.

$$\begin{array}{r} .40547 \\ .69315 \\ \hline \end{array}$$

hyp. log. of 3 = $\frac{.40547}{.69315}$ to five places of decimals.

$$\begin{array}{r} .693147 \\ 2 \\ \hline \end{array}$$

$$\text{hyp. log. } 4 = 1.386294$$

To find the hyperbolic log. of 5 ;

$$\begin{array}{r} 4 \) \ 5.000000 \\ \hline 1.250000 \end{array}$$

$$\begin{aligned}
 &= \downarrow 2, \quad 3, \quad 2, \quad 6, \quad 7, \quad 5, \\
 &= \downarrow 0, \quad 23, \quad \bar{6}, \quad 4, \quad \bar{11}, \quad \bar{5}, \\
 &= \downarrow 0, \quad 0, \quad 224, \quad 4, \quad \bar{103}, \quad \bar{97},
 \end{aligned}$$

$$\begin{array}{r}
 2240 \\
 \quad 4 \\
 \hline
 22440 \\
 \quad 103 \\
 \hline
 223370 \\
 \quad 97 \\
 \hline
 223273
 \end{array}$$

Then $22327 \div 1000574$
 gives $\cdot 22314$ log. of 1.25 ;
 $\log. 4 = \underline{1.38629}$
 $\log. 5 = 1.60943$

In the same manner the hyperbolic logarithms of the remaining consecutive numbers 6, 7, 8, &c., may be determined.

When it is required to find the hyperbolic logarithm of a number greater than 10; suppose all the figures after the first to be decimals, then find the hyperbolic logarithm, and afterwards add 2.3025851 , for two places of figures; 4.6051702 for three places of figures; 6.9077553 for four places of figures, and so on. The following example illustrates this matter.

6. *Required the hyperbolic log. of 345.678.*

First find the hyperbolic log. of 3.45678 .

$$\begin{array}{r}
 3 \) \ 3.45678 \\
 \hline
 1.15226 \\
 \hline
 = \downarrow 1, \quad 4, \quad 6, \quad 6, \quad 1, \quad 7, \\
 = \downarrow 0, \quad 14, \quad 2, \quad 5, \quad \bar{8}, \quad 2, \\
 = \downarrow 0, \quad 0, \quad 142, \quad 5, \quad \bar{64}, \quad \bar{54},
 \end{array}$$

$$\begin{array}{r} 14250 \\ 64 \\ \hline 141860 \\ 54 \\ \hline 141806 \end{array}$$

141806 divided by 1000574 gives the log. of

$$\begin{array}{rcl} 1.15226 & = & .14172 \\ \text{log. of } 3 & = & 1.09861 \\ \hline \text{log. of } 3.45678 & = & 1.24033 \\ \text{log. of } 100 & = & 4.60517 \\ \hline \text{log. of } 345.678 & = & 5.84550 \end{array}$$

In calculating the log. of 345.678 by this method, it was supposed that the log. of 3 was known.

7. *Required the hyp. log. of 345.678, without knowing the log. of any number except the log. of 10 (=2.3025851); the base $\epsilon = 2.718281828$ being also given.*

The square root of 3.45678 may be reduced to $\downarrow 6,4,8,5,1,0,7$, which may be reduced to 6202297, in the seventh position.

Again the square root of 2.718281828 may be reduced to $\downarrow 5,2,3,5,4,9,9$, which may be reduced to 5000488, in the seventh position.

Then $6202297 \div 5000488$ gives 1.240338, hyp. log. of 3.45678.

$$\begin{array}{rcl} & & 1.240338 \\ \text{twice log. } 10 & \dots & 4.605170 \\ \hline \text{hyp. log. } 345.678 & \dots & 5.845508 \end{array}$$

8. *Required the hyperbolic logarithm of 7635.214 by direct calculation to seven places of decimals, having only given the base and the hyperbolic logarithm of 10.*

$$10000 \div 7365.214 = 1.30972098$$

$$\begin{array}{r}
 10\,00\,00\,00\,0 \\
 2\,00\,00\,00\,0 \\
 10\,00\,00\,0 \\
 \hline
 12 \overline{) 10\,00\,00\,0} \dots + 1 \\
 \quad 96 \overline{) 80\,00\,0} \dots + 8 \\
 \quad \quad 3 \overline{) 38\,80\,0} \dots + 28 \\
 \quad \quad \quad 6 \overline{) 77\,0} \dots + 56 \\
 \quad \quad \quad \quad 8 \overline{) 5} \dots + 70 \\
 \quad \quad \quad \quad \quad 1 \dots + 56 \\
 \hline
 13 \, 10 \overline{) 25\,66\,2} \dots + 1 \\
 \quad \quad 52 \, 4 \overline{) 10} \dots - 4 \\
 \quad \quad \quad 1 \overline{) 3} \dots + 10 \\
 \hline
 13 \, 09 \, 73 \overline{) 26\,5} \dots \\
 \quad \quad 1 \overline{) 17\,9} \dots - 9 \\
 \hline
 13 \, 09 \, 72 \, 0 \overline{) 86} \\
 \quad \quad \quad 1 \, 3 \quad + 1
 \end{array}$$

$\therefore 1.30972098 = \downarrow 2,8,0,4,0,9,1$, which is easily reduced to $2698405 = \downarrow 0,0,0,0,0,2698405$,

$\epsilon = \downarrow 10,4,7,1,0,0,4$, when reduced in the same manner becomes 10000978 .

$\therefore 2698405 \div 10000978$ gives $.2698141$, the hyperbolic log. of 1.30972098 ;

$$\begin{array}{rcl}
 \log. \text{ of } 10000 & = & 9.2103404 \\
 \log. \text{ of } 1.30972098 & = & .2698141 \\
 \log. \text{ of } 7365.214 & = & \underline{8.9405263}
 \end{array}$$

9. Required the number corresponding to the hyperbolic logarithm 8.9405263 , by a direct calculation, the log. of $10, = 2.3025851$, being given.

Four times is the first multiple of the log. of 10 , that exceeds the given logarithm,

$$\begin{array}{rcl}
 4 \text{ times } 2.3025851 & = & 9.2103404 \\
 \text{given log.} & & 8.9405263 \\
 \hline
 & & .2698141
 \end{array}$$

$$\epsilon = \downarrow 10, 4, 7, 1, 0, 0, 4, \\ = \downarrow 0, 0, 0, 0, 0, 0, 10000978,$$

$\cdot 2698141 \times 10000978 = 2698405$, the decimal part being rejected.

$$\begin{aligned} \text{Then } & \downarrow 0, 0, 0, 0, 0, 0, 2698405, \\ & = \downarrow 2, 0, 0, 0, 0, 0, 792015, \\ & = \downarrow 2, 7, 0, 0, 0, 0, 95424, \\ & = \downarrow 2, 7, 9, 0, 0, 0, 5460, \\ & = \downarrow 2, 7, 9, 5, 4, 6, 0, \end{aligned}$$

This reducing is simple enough, since

$$\begin{aligned} \downarrow 1, & = \downarrow 0, 10, \bar{4}, \bar{1}, \bar{9}, \bar{5}, \bar{1}, \\ & = \downarrow 0, 0, 0, 0, 0, 0, 953195, \\ \downarrow 0,1, & = \downarrow 0, 0, 10, 0, \bar{4}, \bar{4}, \bar{7}, \\ & = \downarrow 0, 0, 0, 0, 0, 0, 99513, \\ \downarrow 0,0,1, & = \downarrow 0, 0, 0, 10, 0, 0, \bar{4}, \\ & = \downarrow 0, 0, 0, 0, 0, 0, 9996, \end{aligned}$$

$$\begin{array}{r} 953195 \) \ 2698405 \ (\ 2 \\ \underline{1906390} \end{array}$$

$$\begin{array}{r} 99513 \) \ 792015 \ (\ 7 \\ \underline{696591} \end{array}$$

$$\begin{array}{r} 9996 \) \ 95424 \ (\ 9 \\ \underline{89964} \end{array}$$

$$5460$$

In this way the resulting numbers are found, and afterwards arranged in the usual form.

$$\downarrow 2,7,9,5,4,6,0,$$

It is evident that $\downarrow 2,0,0,0,0,0,0$, must be added when $\downarrow 0,0,0,0,0,0,190639$, is subtracted; that $\downarrow 0,7,0,0,0,0,0$, must

be added when $\downarrow 0,0,0,0,0,696591$, is subtracted; and that $0,0,9,0,0,0,0$, must be added when $\downarrow 0,0,0,0,0,89964$, is subtracted.

$$\begin{array}{r}
 1000000000 \\
 2000000000 \\
 1000000000 \\
 \hline
 12 \overline{) 1000000000} \quad \sim 1 \\
 \quad 847000000 \quad \sim 7 \\
 \quad \quad 25410000 \quad \sim 21 \\
 \quad \quad \quad 4235 \quad \sim 35 \\
 \quad \quad \quad \quad 42 \quad \sim 35 \\
 \hline
 129 \overline{) 728377} \quad \sim 1 \\
 \quad 1167555 \quad \sim 9 \\
 \quad \quad 4670 \quad \sim 36 \\
 \quad \quad \quad 11 \quad \sim 84 \\
 \hline
 1309 \overline{) 00613} \dots \sim 1 \\
 \quad \quad 64450 \dots \sim 5 \\
 \quad \quad \quad 13 \dots \sim 10 \\
 \hline
 13096 \overline{) 6076} \dots \\
 \quad \quad 5238 \dots \sim 4 \\
 \quad \quad \quad 786 \dots \sim 6 \\
 \hline
 130972100
 \end{array}$$

$\therefore$ 10000 divided by 1.309721 gives 7365.214, the number required.

10. Required the number corresponding to the hyperbolic logarithm of 2.1972245.

It is evident that the required number must be less than 10.

$$\begin{array}{r}
 \text{From } 2.3025851 \\
 \text{take } 2.1972245 \\
 \hline
 \end{array}$$

$$.1053606 = \log \frac{10}{x},$$

x being the required number.

$$.1053606 \times 10000978 = 1053709, \text{ neglecting decimals.}$$

$$\begin{array}{r} 953195 \) \ 1053709 \ \downarrow 1, \\ \underline{953195} \end{array}$$

$$\begin{array}{r} 99513 \) \ 100514 \ (\ 1, \\ \underline{99513} \end{array}$$

$$9996 \) \ 1001 \ (\ 0,$$

Find the number corresponding to

$$\downarrow 1,1,0,1,0,0,1,$$

$$10 \ 00 \ 00 \ 00 \ 0$$

$$\underline{1 \ 00 \ 00 \ 00 \ 0}$$

$$\begin{array}{r} 11 \ | \ 00 \ | \ 00 \ | \ 00 \ | \ 0 \ . \\ \underline{11 \ | \ 00 \ | \ 00 \ | \ 0 \ .} \end{array}$$

$$\begin{array}{r} 11 \ | \ 11 \ 00 \ 00 \ | \ 0 \ . \ . \ . \\ \underline{11 \ | \ 11 \ 00 \ 00 \ | \ 0 \ . \ . \ .} \end{array}$$

$$\begin{array}{r} 11 \ 11 \ 11 \ 11 \ | \ 10 \ . \ . \ . \ . \ . \\ \underline{11 \ 11 \ 11 \ 11 \ | \ 10 \ . \ . \ . \ . \ .} \end{array}$$

$$11 \ 11 \ 11 \ 11 \ 2$$

$\therefore$ 10 divided by 1.1111112 gives 9 very nearly.

$\therefore$ 2.1972245 is the hyp. log. of 9.

the reciprocal with respect to 10.

$$4 \) \ 2.1972245$$

.5493061 log. of fourth root of the required number.

$$5493061 \times 10000978 = 5493598.$$

$$\begin{array}{r} 953195 \) \ 5493598 \ \downarrow 5, \\ \underline{4765975} \end{array}$$

$$\begin{array}{r} 99513 \) \ 727623 \ (\ 7, \\ \underline{696591} \end{array}$$

$$\begin{array}{r} 9996 \) \ 31032 \ (\ 3 \\ \underline{29988} \end{array}$$

$$1044$$

$\therefore \downarrow 5,7,3,1,0,4,4$, represents the fourth root of the required number.

$$\begin{array}{r}
 \begin{array}{r|rrrrrrrr}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & 5 & 0 & 0 & 0 & \\
 & & & & 1 & 0 & 0 & \\
 \hline
 1 & 6 & 1 & 0 & 5 & 1 & 0 & 0 \\
 & 1 & 1 & 2 & 7 & 3 & 5 & 7 \\
 & & 3 & 3 & 8 & 2 & 1 & \\
 & & & 5 & 6 & 4 & & \\
 & & & & 6 & & & \\
 \hline
 1 & 7 & 2 & 6 & 6 & 8 & 4 & 8 \\
 & & 5 & 1 & 8 & 0 & 1 & \\
 & & & 5 & 2 & & & \\
 \hline
 1 & 7 & 3 & 1 & 8 & 7 & 0 & 1 \\
 & & 1 & 7 & 3 & 2 & & \dots \\
 & & & 6 & 9 & & & \dots \\
 & & & & 7 & & & \dots \\
 \hline
 1 & 7 & 3 & 2 & 0 & 5 & 0 & 9
 \end{array}
 \end{array}$$

$\therefore 1.7320509$ is the fourth root of the required number, but 1.7320509 is the square root of 3, hence 9 is the required number.

In works on geometry, methods are sometimes given to find distances of inaccessible objects without the aid of instruments to measure angles; but such rules and directions can seldom be applied for want of sufficient suitable space to operate upon. It mostly happens that the distance required is great compared with the convenient base from which observations have to be made, besides the angles at the base approach right angles, one of them often greater than a right angle. The next problem shows how to find inaccessible distances with great accuracy from confined bases, the means of measuring a straight line being only required.

11. An observer at A wishes to find the inaccessible distance AC : the base AB measures 8 chains; AD in a direct line to C measures 10 chains, and BE 14; then DB is found by measurement to be 11 chains, and AE 17; required all the angles of the triangle ACB and the distance AC .

Fig. 11.

$$\cos DAB = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{100 + 64 - 121}{160} = .26875000$$

$$\cos ABE = \frac{196 + 64 - 289}{224}$$

$$= - .13660714$$

The cosine of the angle ABE being negative, indicates the angle to be greater than a right angle.

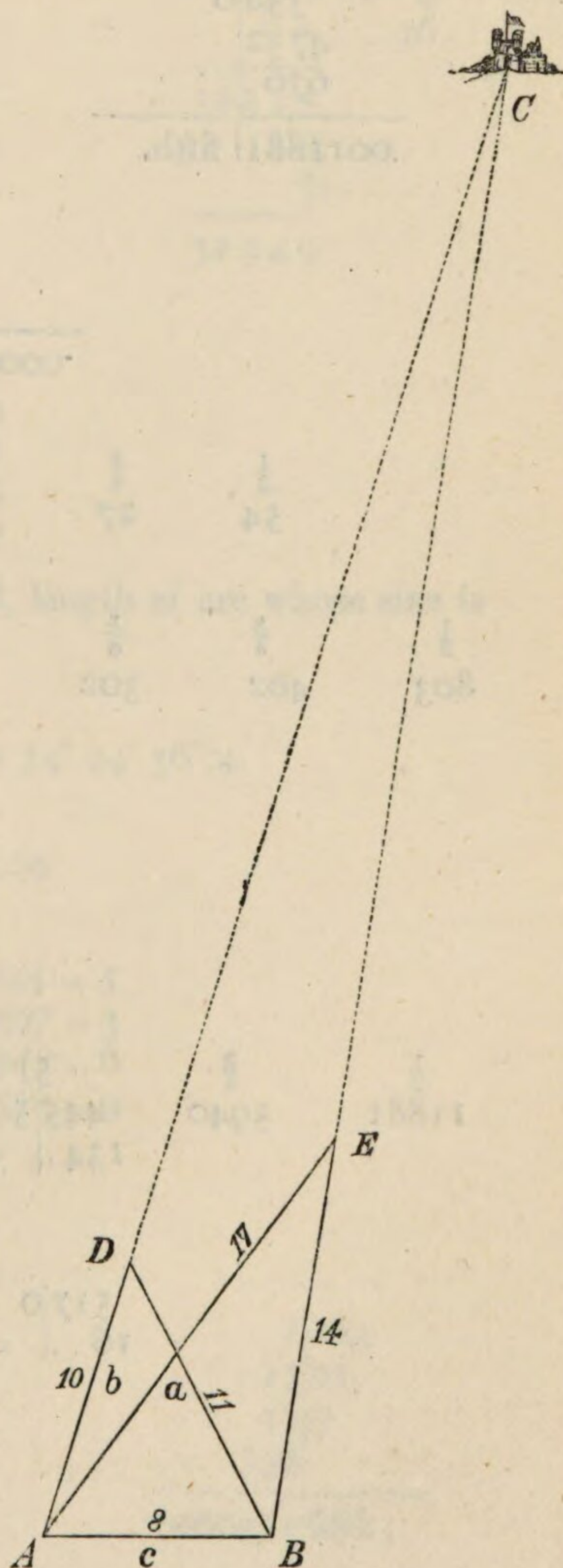
$$.26875000 = .26 \downarrow 0,3,3,2,5,0,0,5,$$

$$.13660714 = .13 \downarrow 0,5,0,2,2,3,1,$$

$$(.26)^2 = .0676$$

1 ~ 676—	3380 ~ 5
2 ~ 1352	2028 ~ 3
4 ~ 2704	4056 ~ 6
8 ~ 5408	6084 ~ 9
676	
7 ~ 4732	

It is easy to find 1, 2, 3, 4, 5, 6, 7, 8, and 9 times 676; then the required powers of (.26) are obtained by mere inspection.



4056

1352

$$\cdot 26^3 = \cdot 017576$$

$$\begin{array}{r|l} 4056 & \\ 4732 & \\ 3380 & \\ 4732 & \\ 676 & \\ \hline .0011881 & \text{fifth.} \end{array}$$

$$\begin{array}{r|l} 676 & \\ 5408 & \\ 5408 & \\ 676 & \\ 676 & \\ \hline .0000803 & \text{seventh.} \end{array}$$

$$\begin{array}{r|l} 2028 & \\ 54080 & \\ \hline .0000054 & \text{ninth.} \end{array}$$

$$\frac{1}{2} \quad 54$$

$$\frac{3}{4} \quad 27$$

$$\frac{5}{6} \quad 20$$

$$\frac{7}{8} \quad 17$$

$$9) \quad 15 \quad 2$$

$$\frac{1}{2} \quad 803$$

$$\frac{3}{4} \quad 402$$

$$\frac{5}{6} \quad 302$$

$$7) \quad \begin{array}{l} 252 \\ 8 \dots \end{array} \sim 3 \sim$$

$$\downarrow 0, 21, 21,$$

$$\begin{array}{l} 36 \dots \sim 1 \\ 8 \dots \sim 21 = A \\ 1 \dots \sim A \times 20 \div 2 \end{array}$$

$$\begin{array}{l} 315 \dots \\ 9 \dots \end{array} \sim 3 \sim$$

$$\begin{array}{l} 45 \dots \\ 1 \dots \\ \hline 46 \end{array}$$

$$\frac{1}{2} \quad 11881$$

$$\frac{3}{4} \quad 5940$$

$$5) \quad \begin{array}{l} 445 \\ 134 \dots \end{array} \sim 3 \sim$$

$$\downarrow 0, 15, 15, 10, \text{ \&c.}$$

$$\begin{array}{l} 89 \dots \sim 1 \\ 13 \dots \sim 15 = A \\ 9 \dots \sim A \times 14 \div 2 = B \\ \dots \sim B \times 13 \div 3 \end{array}$$

$$\begin{array}{l} 517 \dots \\ 16 \dots \end{array} \sim 3 \sim$$

$$\begin{array}{l} 103 \dots \\ 1 \dots \end{array}$$

$$\begin{array}{l} 105 \dots \\ 1 \dots \end{array}$$

$$105 \ 1$$

$$\frac{1}{2}$$

$$175760$$

$$3)$$

$$87880$$

$$\downarrow 0,9,9,6,15,0,$$

$$\begin{array}{r|l} 29 & 29 \ 3 \ . \sim 1 \\ 2 & 63 \ 6 \ . \sim 9 \\ & 10 \ 5 \ . \sim 36 \\ & 2 \ . \sim 84 \\ \hline 32 & 0 \ 3 \ 6 \ . \sim 1 \\ & 2 \ 8 \ 8 \ . \sim 9 \\ & 1 \ . \sim 36 \\ \hline 32 & 3 \ 2 \ 5 \ . \dots \\ & 1 \ 9 \ . \dots \\ & 5 \ . \dots \\ \hline 32 & 3 \ 4 \ 9 \end{array}$$

$$\begin{array}{r} .2687500 \\ 32349 \\ 1051 \\ 46 \\ 2 \\ \hline \end{array}$$

$\cdot 2720948$, length of arc whose sine is

$$\cdot 2687500 \quad \therefore \text{arc} = 15^\circ 35' 23'' \cdot 6$$

$$\therefore \text{the angle } CAB = 74^\circ 24' 36'' \cdot 4.$$

$$(\cdot 13)^2 = \cdot 0169$$

$$\begin{array}{ll} 1 \sim 169 & 845 \sim 5 \\ 2 \sim 338 & 507 \sim 3 \\ 4 \sim 676 & 1014 \sim 6 \\ 8 \sim 1352 & 1521 \sim 9 \\ & 169 \\ 7 \sim 1183 \end{array}$$

$$\begin{array}{r} 507 \\ 169 \\ \hline \cdot 0021970 \text{ cube;} \end{array}$$

$$\begin{array}{r|l} 1 & 183 \\ 15 & 21 \\ 16 & 9 \\ 338 & \\ \hline \cdot 0000371 \text{ fifth;} \end{array}$$

$$\begin{array}{r|l} & 169 \\ 1 & 183 \\ 5 & 07 \\ \hline \end{array}$$

$\cdot 0000006$ seventh; which may be neglected, as it will not, when reduced, give a unit in the seventh decimal place.

$$\frac{1}{2}$$

$$371$$

$$\frac{3}{4}$$

$$186$$

$$5)$$

$$140$$

$$7 \dots \sim 5 \sim$$

$$\downarrow 0,25,0,\overline{10}, \&c.$$

$$28 \dots \sim 1$$

$$7 \dots \sim 25 = A$$

$$1 \dots \sim A \times 24 \div 2$$

$$36$$

$$21970$$

$$3)$$

$$10985$$

$$549 \dots \sim 5 \sim$$

$$\downarrow 0,15,0,\overline{6},6, \&c.$$

$$36 \dots \sim 1$$

$$5 \dots \sim 15 = A$$

$$38 \dots \sim A \times 14 \times 2 = B$$

$$2 \dots \sim B \times 13 \div 3$$

$$42 \ 51 \dots$$

$$3 \dots \text{negative 6 times.}$$

$$42 \ 48$$

$$1366071$$

$$4248$$

$$36$$

$$\cdot 1370355, \text{ length of arc whose sine is } \cdot 1366071.$$

$$\therefore \text{ arc} = 7^\circ 51' 5'' \cdot 6$$

$$\therefore \text{ angle } ABC = 97^\circ 51' 5'' \cdot 6, \text{ since its cosine is negative.}$$

$$\therefore \text{ The angle } ACB = 7^\circ 44' 18''.$$

$$\text{From } \cdot 2720948$$

$$\text{take } \cdot 1370355$$

$$\cdot 13 \downarrow 0,4,\overline{2},3, = \cdot 1350593 \text{ length of arc of } 7^\circ 44' 18''.$$

$$(\cdot 13)^3 = \cdot 0021970$$

$$(\cdot 13)^5 = \cdot 0000371$$

$$\begin{array}{r} 2) \\ 371 \end{array}$$

$$\begin{array}{r} 3) \\ 186 \end{array}$$

$$\begin{array}{r} 4) \\ 62 \end{array}$$

$$\begin{array}{r} 5) \\ 15. \\ 6.. \end{array} \sim 4 \sim$$

$$\downarrow 0,20, \&c. \\ \begin{array}{r} 3. \\ 1. \\ \hline 4 \end{array}$$

$$\begin{array}{r} 2 \\ 21970 \end{array}$$

$$\begin{array}{r} 3 \\ 109 \end{array} 85 \sim 4 \sim$$

$$\begin{array}{l} \downarrow 0,12,6,9, \\ 36 \overline{) 62} \sim 1 \\ 4 \overline{) 40} \sim 12 = A \\ 24 \sim A \times 11 \div 2 \\ \hline 412 \overline{) 6} \dots \\ 2 \overline{) 5} \dots \sim 6 \text{ minus} \\ \hline 4101 \dots \\ 4 \dots \\ \hline 4105 \end{array}$$

$$\begin{array}{r} 1350593 + \\ 4105 - \\ \hline 1346488 \\ 4 + \\ \hline 1346492 \text{ sine of } 7^\circ 44' 18''. \end{array}$$

$\therefore \sin 7^\circ 44' 18'' : 8 :: \sin 97^\circ 51' 5''.6 : AC$, the distance required.

$$\{1 - (.13660714)^2\}^{\frac{1}{2}} = .9906244 = \sin 97^\circ 51' 5''.6.$$

$$\therefore .1346492 : 8 :: .9906244 : 58.864 \text{ the required distance.}$$

In solving this example, lengthened details are entered into, to prevent obscurity; the results are carried to a degree of accuracy seldom required in practice, to show the delicacy and powers of the method. It may also be observed, that, to find the distance it was not necessary to find the degrees, minutes, &c. contained in the arcs.

The work of this example, without superfluities, carried out to five places of decimals, may be arranged as follows:

$$\frac{100 + 64 - 121}{160} = \cdot 26875 = \cos CAB = \cdot 26 \downarrow 0,3,3,2,5,$$

$$\frac{196 + 64 - 289}{224} = - \cdot 13661 = \cos CBA = - \cdot 13 \downarrow 0,5,0,\bar{2},2,$$

$$(\cdot 26)^5 = \cdot 01758$$

$$(\cdot 26)^5 = \cdot 00119$$

$$\frac{1}{2} \\ 119$$

$$\frac{3}{4} \\ 60$$

$$\begin{array}{r} 5) \\ 45 \\ 14 \cdot \end{array} \sim 3 \sim$$

$$\begin{array}{r} \downarrow 0,15,15 \\ 9 \cdot \dots \\ 1 \cdot \dots \\ \hline 10 \end{array}$$

$$\frac{1}{2} \\ 1758$$

$$\begin{array}{r} 3) \\ 879 \end{array}$$

$$\begin{array}{r} \downarrow 0,9,9, \\ 29 \overline{) 3 \cdot} \\ 26 \cdot \\ \hline 1 \cdot \\ 320 \overline{) \dots} \\ 3 \cdot \dots \\ \hline 323 \end{array}$$

$$\begin{array}{r} \cdot 26875 \\ 323 \\ \hline 10 \end{array}$$

$$\text{arc} = \cdot 27208 \text{ to } \sin 26875; \cos CAB.$$

$$(\cdot 13)^3 = \cdot 00220$$

$$\frac{1}{2} \\ 220$$

$$\begin{array}{r} 3) \\ 110 \\ 6 \cdot \cdot \sim 5 \sim \end{array}$$

$$\begin{array}{r} \downarrow 0,15,0, \\ 37 \overline{) \dots} \\ 6 \cdot \\ \hline 43 \end{array}$$

$$\begin{array}{r} \cdot 13661 \\ 43 \\ \hline \end{array}$$

$$\text{arc} = \cdot 13704 \text{ to } \sin \cdot 13661; \cos CBA$$

$$\begin{array}{r} \cdot 27208 \\ \cdot 13704 \\ \hline \end{array}$$

$$\cdot 13 \downarrow 0,4,\bar{2}, = \cdot 13504 = \text{arc of angle } ABC.$$

$$(\cdot 13)^3 = \cdot 00220$$

2)
220

3
110

↓ 0,12,6,
37 | ..
4 | ..
—
41

·13504
41
—
·13463 sine of arc ·13504.

∴ ·13463 : 8 :: 99062 : 58·864 as before.

The length of an arc (·13704) to radius 1, and its sine (·13661) being known, and the cosine required: it is a question whether it is easier to find the cosine from the sine by the well known expression,

$$\sqrt{1 - \sin^2} = \cos,$$

or to find the cosine, by Dual Arithmetic, in a direct manner from the length of the arc. To square five or six figures, and afterwards to extract the square root of twice as many, appear to involve more mental labour than the following direct method:

·13704 = ·13 ↓ 0,5,3,
(·13)² = ·01690 to five decimal places
(·13)⁴ = ·00029.

2)
29

3)
15

4)
5

1

2)
1690

↓ 0,10,6,
84 | 5 .
8 | 5 . ∪ 10 = A
4 | 4 . ∪ A × 9 ÷ 2
—
93 4 | ...
5 | ...
—
93 9

1·00000
939 —
—
·99061
1 +
—
·99062 = cosine.

12. Find the meridional parts answering to any given latitude ($35^\circ 0'$) by a direct calculation.

The meridional parts y for any latitude x , is given by the formula

$$y = 3437.74679 \log. \tan (45^\circ + \frac{1}{2} x).$$

Hyperbolic logarithms are employed, and 3437.74679 are the nautical miles or minutes that in every circle is equal to its radius.

$$\begin{array}{r} 90^\circ 0' \\ 35 \quad 0 \\ \hline 2 \quad) \quad 125 \quad 0 \end{array}$$

$$\cos 27^\circ 30' = \sin 62^\circ 30'$$

Length of arc of $27^\circ 30' = .4799655$.

$$.4799655 = .48 \downarrow 0,0,0,0,\overline{7},\overline{2},$$

$(.48)^2 = .2304000$. When necessary, the seven places.

$$(.48)^3 = .1105920$$

$$(.48)^4 = .0530842$$

$$(.48)^5 = .0254804$$

$$(.48)^6 = .0122306$$

$$(.48)^7 = .0058707$$

$$\begin{array}{cccccc} 2) & 3) & 4) & 5) & 6) & 7) \\ 58707 & 29354 & 9785 & 2446 & 489 & 82 & 12. \end{array}$$

$$\begin{array}{cccccc} 2) & 3) & 4) & 5) & 6) & \\ 122306 & 61153 & 20384 & 5096 & 1019 & 170. \end{array}$$

$$\begin{array}{cccccc} 2) & 3) & 4) & 5) & \downarrow 0,0,0,0,\overline{35}, & \\ 254804 & 127402 & 42467 & 10617. & 2124. & \dots \dots \dots \\ & & & 7 \dots \dots & 1. & \dots \dots \dots \text{minus} \\ & & & & \hline & & & & 2123 \end{array}$$

$$\begin{array}{cccc} 2) & 3) & 4) & \downarrow 0,0,0,0,\overline{35},\overline{8}, \\ 530842 & 265421 & 88474 & 22119 \dots \dots \dots \\ & & 6 \dots \dots & 6 \dots \dots \dots \\ & & & \hline & & & 22113 \end{array}$$

$$\begin{array}{r}
 2) \\
 1105920 \\
 \hline
 \end{array}
 \qquad
 \begin{array}{r}
 3 \\
 55296 \overline{)0} \\
 39 \dots \sim 7 \sim
 \end{array}
 \qquad
 \begin{array}{r}
 \downarrow 0,0,0,0,\overline{21,6}, \\
 18432 \overline{)0} \dots \\
 3 \overline{)9} \dots \text{minus} \\
 \hline
 18428 \overline{)1} \dots \\
 1 \overline{)1} \dots \text{minus} \\
 \hline
 184280
 \end{array}$$

$$\begin{array}{r}
 2 \\
 23040 \overline{)00} \\
 161 \dots \sim 7 \sim
 \end{array}
 \qquad
 \begin{array}{r}
 \downarrow 0,0,0,0,\overline{14,4}, \\
 11520 \overline{)00} \dots \\
 1 \overline{)61} \dots \text{minus} \\
 \hline
 11518 \overline{)3} \overline{)9} \dots \\
 5 \dots \text{minus} \\
 \hline
 1151834
 \end{array}$$

$$\begin{array}{r}
 1.0000000 \\
 .1151834 - \\
 \hline
 .8848166 \\
 22113 + \\
 \hline
 .8870279 \\
 170 - \\
 \hline
 .8870109 = \cosine\ 27^\circ\ 30'
 \end{array}
 \qquad
 \begin{array}{r}
 .4799655 \\
 184280 - \\
 \hline
 .4615375 \\
 2123 + \\
 \hline
 .4617498 \\
 12 \\
 \hline
 .4617486 = \sin\ 27^\circ\ 30'.
 \end{array}$$

Now .8870109 is the sine of $62^\circ\ 30'$, and .4617486 is the cosine, and since the sine divided by the cosine gives the tangent we may proceed as follows to find the hyperbolic log. of the tangent.

$$\begin{array}{l}
 .8870109 \text{ multiplied by } \downarrow 1,2,4,6,8,9,1, = 1 \\
 .4617486 \text{ multiplied by } \downarrow 8,1,0,3,0,3,0, = 1
 \end{array}$$

$\therefore$.8870109 divided by .461748, will be represented by

$$\begin{array}{r}
 \downarrow 8, \ 1, \ 0, \ 3, \ 0, \ 3, \ 0, \\
 \text{Divided by } \downarrow 1, \ 2, \ 4, \ 6, \ 8, \ 9, \ 1, \\
 \hline
 \downarrow 7, \ \overline{1}, \ \overline{4}, \ \overline{3}, \ \overline{8}, \ \overline{6}, \ \overline{1}, \text{ represents the quotient,}
 \end{array}$$

which is readily reduced to 6529007, which, when divided by 10000978, gives .6528368, the hyperbolic logarithm of the tangent of $62^{\circ} 30'$. Then $3437.74679 \times .6528368 = 2244.287$, the meridional parts required.

The multiplication of any given number by 3437.74679, is a matter of mere inspection when 1, 2, 3, 4, 5, 6, 7, 8, 9, times 3437.74679 are first set down.

WORK WITHOUT CONTRACTION.

$$\begin{array}{rcl}
 1 \sim 3437.74679 & \text{---} & 17188.73395 \sim 5 \\
 2 \sim 6875.49358 & \text{---} & 10313.24037 \sim 3 \\
 4 \sim 13750.98716 & & 20626.48074 \sim 6 \\
 8 \sim 27501.96432 & & 30939.72111 \sim 9 \\
 & & 3437.74679 \\
 7 \sim 24064.22753
 \end{array}$$

$$\begin{array}{r}
 2 \mid 750197432 \\
 20 \mid 62648074 \\
 103 \mid 1324037 \\
 2 \mid 750197432 \\
 6 \mid 87549358 \\
 171 \mid 8873395 \\
 2062 \mid 648074 \\
 \hline
 2244.287 \mid
 \end{array}$$

This example is carried to an extent seldom required in practice. The whole theory of Mercator's sailing depends upon the accuracy of these meridional parts.

Since $\frac{10000}{3} \downarrow 0,3,1$, represents 3437.74679 nearly, then $\frac{10000}{3} \times .6528368 \downarrow 0,3,1$, gives the meridional part nearly.

WORK.

$$\begin{array}{r} 65 \overline{) 28368} \\ \underline{195} \\ 1959 \\ \underline{ 7} \end{array}$$

$$\begin{array}{ccc|ccc} 6 & 7 & 2 & 6 & 1 & 8 & 5 & . & . \\ & & & 6 & 7 & 2 & 6 & . & . \end{array}$$

3) 6732.911

22 44'3

meridional parts, lat. 35° ,

and near enough the truth for practice.

13. Find without the use of tables of logarithms, or of sines and tangents, the true meridional parts corresponding to 80° of latitude, not the approximate meridional parts given by Mr. Wright.

$$\begin{array}{r} 90^{\circ} 0' \\ 80 \quad 0 \\ \hline 2 \overline{) 170 \quad 0} \\ \hline 85^{\circ} 0' \end{array}$$

The cosine of $5^\circ = \text{sine of } 85^\circ$.

The length of arc of $5^\circ = \cdot 087266$.

By the method of calculation so fully explained and illustrated, the cosine of this arc is found to be $\cdot 996195$, and the sine $= \cdot 0871557$. The process is so simple for an angle so small, that it is unnecessary to insert the work.

$$\begin{array}{c|c|c|c|c} 8 & 7 & 1 & 5 & 7 \\ 8 & 7 & 1 & 5 & 6 \end{array} \quad \downarrow 1, 4, 2, 3, 6, 3,$$

$$\begin{array}{r|l} 95 & 8713 \dots \\ 3 & 8349 \dots \\ & 575 \dots \\ & 4 \dots \end{array}$$

997641 . . .

↓ 0,0,3,8,1,3,

$$\begin{array}{r|l} 996 & 195 \\ 2 & 989 \\ & 3 \end{array} \quad \begin{array}{l} \dots \\ \dots \\ \dots \end{array}$$

$$\begin{array}{r} 999 \text{ I } 87 \dots \\ 800 \dots \end{array}$$

999 9 8 7

$$\begin{array}{r} 997 \overline{641} \dots \\ 1995 \dots \\ 1 \dots \end{array}$$

$$\begin{array}{r} 99998 \overline{7} \dots \\ 10 \dots \\ 3 \dots \end{array}$$

$$\begin{array}{r} 9996 \overline{37} \dots \\ 300 \dots \\ 60 \dots \\ 3 \dots \end{array}$$

$$\therefore \cdot 996195 \times \downarrow 0,0,3,8,1,3, = 1.$$

$$\text{and } \cdot 0871557 \times \downarrow 1,4,2,3,6,3, = \frac{1}{10}.$$

$$\therefore \cdot 996195 \text{ is represented by } \frac{1}{\downarrow 0,0,3,8,1,3,}$$

$$\text{and } \cdot 0871557 \text{ by } \frac{1}{10 \downarrow 1,4,2,3,6,3,}$$

$$\frac{\cdot 996195}{\cdot 0871557} = \frac{10 \downarrow 1,4,2,3,6,3,}{\downarrow 0,0,3,8,1,3,} = 10 \downarrow 1,4,1,5,5,0,$$

To render this method clear, every step is set down, figure by figure.

$$\begin{aligned} & \downarrow 1, \quad 4, \quad \overline{1}, \quad \overline{5}, \quad 5, \quad 0, \\ & = \downarrow 0, \quad 14, \quad \overline{5}, \quad \overline{6}, \quad \overline{4}, \quad \overline{5}, \\ & = \downarrow 0, \quad 0, \quad 135, \quad \overline{6}, \quad \overline{60}, \quad \overline{75}, \end{aligned}$$

$$\begin{array}{r} 1350 \\ 6 \end{array}$$

$$\begin{array}{r} 13440 \\ 60 \end{array}$$

$$\begin{array}{r} 133800 \\ 75 \end{array}$$

$$1000470 \overline{) 133725} \text{ (quotient } \cdot 13366 \text{ } \frac{2 \cdot 30259}{\text{hyp. log. } 10} \text{)}$$

$$\text{Hyp. log. tan } (45^\circ + \text{half lat.}) = 2 \cdot 43625$$

This result must be multiplied by

$$\frac{10000}{3} \downarrow 0,3,1,$$

WORK.

$$\begin{array}{r}
 24 \overline{) 3625} \\
 \underline{7309} \\
 73 \\
 \hline
 251 \overline{) 007} \\
 \underline{251} \\
 3 \overline{) 251258} \\
 \underline{83752} \text{ meridional parts for lati-}
 \end{array}$$

tude 80° . The number $3437.74679 = \frac{180 \times 60}{3.14159265 \dots}$; because
 $\text{rad. of earth} \times 3.14159265 \dots = \text{length of an arc of } 180^\circ, = \text{length}$
of arc of 10800 minutes or nautical miles.

$$\therefore \text{Rad. of earth} = \frac{10800}{3.14159265} = 3437.74679 \text{ nautical miles.}$$

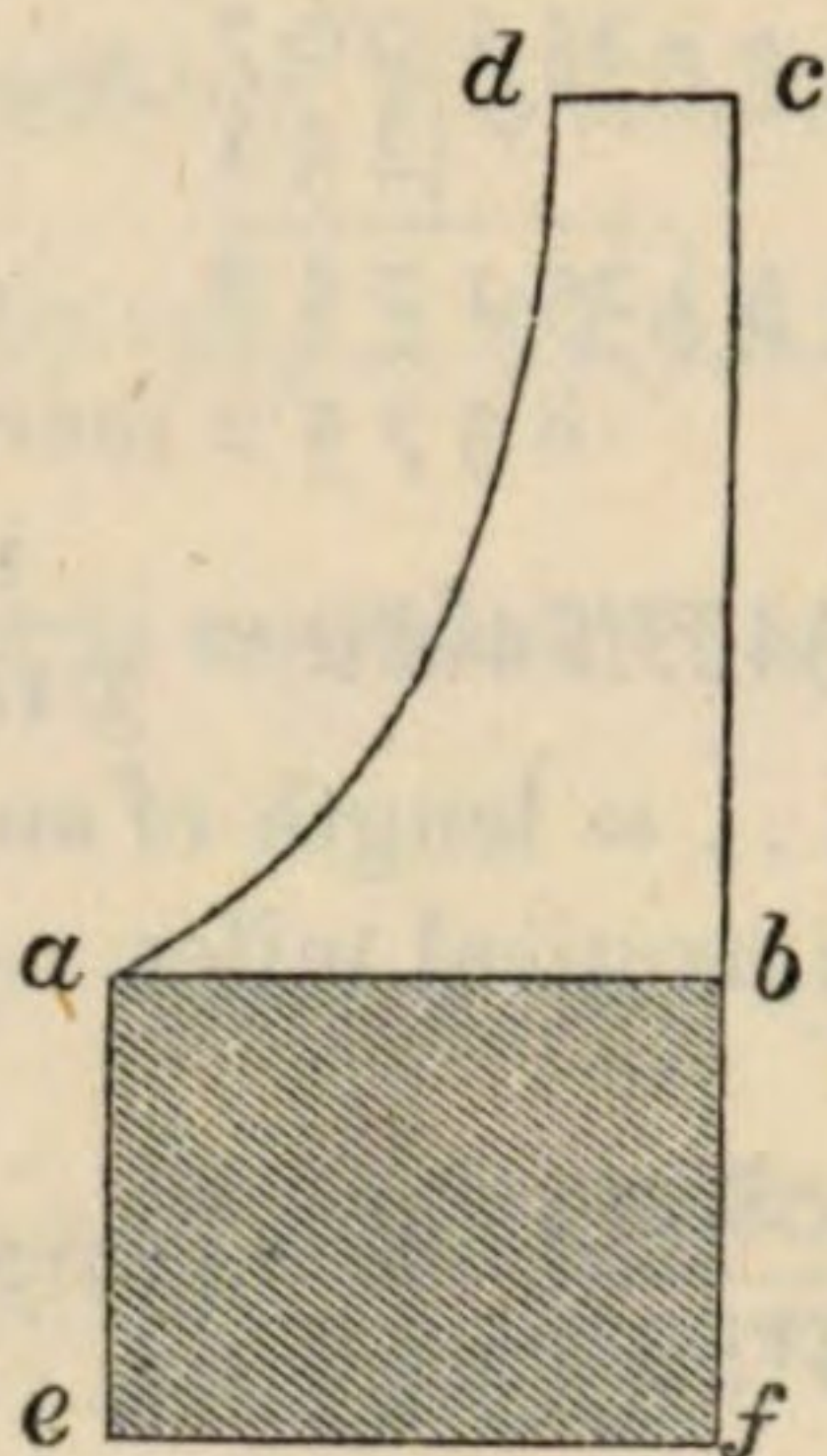
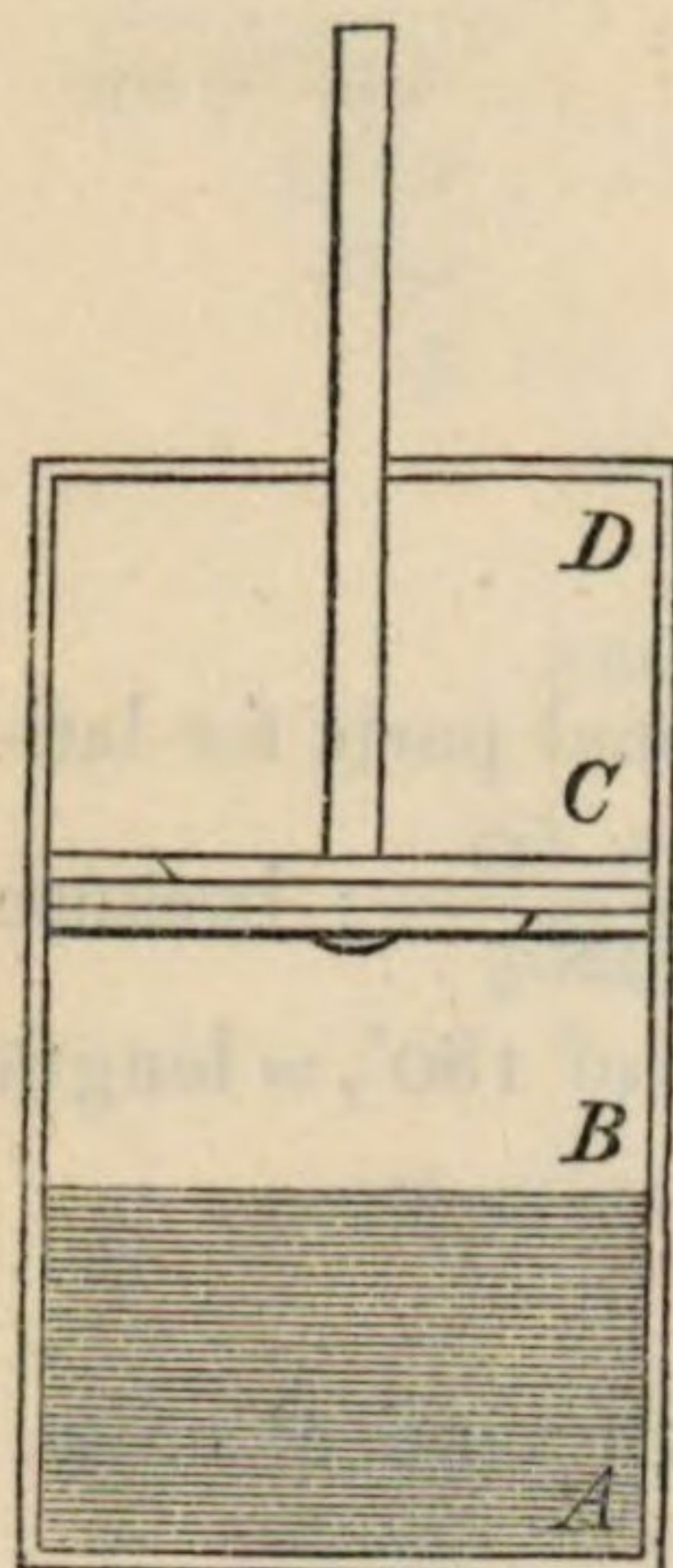
14. *The pressure of steam upon the piston is 65 lbs. to the square inch, the length of the stroke = 11.6 feet, the steam is cut off when 2.4 feet of the stroke is made; find the number of units of work done upon each square inch of the piston. The hyperbolic log. of 10 = 2.3025851, and the base of the system ($\epsilon = 2.718281828$) $\epsilon = \downarrow 10,4,7,1,0,0, = \downarrow 0,0,0,0,0,1000470$, being given.*

RULE GIVEN BY WRITERS ON THE STEAM ENGINE.

Multiply the pressure at which the steam is admitted by the distance travelled by the piston before the steam is cut off; this gives the work done before expansion begins. Divide the whole length of the stroke by the above-mentioned distance, and find the hyperbolic logarithm of the quotient. Multiply this hyperbolic log. by the work done before expansion begins. Adding the work done after to that done before expansion, the whole work done upon one square inch of the piston in one stroke will be obtained.

Investigation.

Fig. 12.



Let $x = AC$, the number of feet described at any part of the stroke, p_2 the pressure when that part is described; $a = AB$, the number of feet described before the steam is cut off, and $l = AD$, the length of the stroke in feet; then let p_1 be the pressure at which the steam is admitted; by Boyle's law,

$$p_2 : p_1 :: a : x$$

$$\therefore p_2 x = p_1 a,$$

that is, any height AC multiplied by the pressure at C , is equal to any height AB multiplied by the pressure at B ; or, $p_2 = \frac{p_1 a}{x}$; the variable work equals the integral of $p_2 dx$ taken between the limits, $x = a$ and $x = l$, or

$$\int_a^l p_2 dx = \int_a^l p_1 a \frac{dx}{x} = ap_1 (\log. l - \log. a) = ap_1 \log. \frac{l}{a}.$$

This is the work done on the square inch during expansion; the work done before expansion is evidently represented by ap_1 . Hence the whole work done on the square inch

$$= ap_1 + ap_1 \log \frac{l}{a};$$

therefore the rule is established.

Calculation.

It is desirable, to solve this question in the most independent way, and to this end, first calculate the hyperbolic log. of 2.

$$2^{10} = 1024$$

$$1.024 = \downarrow 0,2,3,8,1,7, = \downarrow 0,0,0,0,0,23727$$

23727 divided by 1000470 gives .023716, the hyp. log. of 1.024.

$$\cdot 023716$$

6.907754 three times the log of 10,

log. of 1024 = 6.931470 divided by 10, gives .693147, the log. of 2.

11.6 divided by 2.4, gives 4.833333, the hyp. log. of which has to be found.

$$2 \) \ 4.833333$$

$$2 \) \ 2.416666$$

$$1.208888 = \downarrow 2,0,\bar{1},0,8,0,$$

$$= \downarrow 0,0,0,0,0,189790,$$

∴ 189790 divided by 1000470, gives .189701, the hyp. log. of 1.208888.

$$\cdot 189701$$

1.386294 twice log. 2

$$\text{log. of } 4.833333 = 1.575995$$

$$65 \times 2.4 = 156 \text{ units of work}$$

$$1.576 \times 156 = 245.856$$

401.856 units of work required.

The work done on one square inch of the piston is sometimes termed the *load*; in the present case the load will be

$$\frac{401.856}{11.6} = 34.64 \text{ lbs.}$$

Suppose the cylinder to be 88 inches in diameter, then the area = 6082 square inches. If the piston makes 16 double strokes a minute, that is, 16 revolutions of the crank, the horse-power of the engine =

$$\frac{6082 \times 32 \times 401.856}{33000} = 2370 \text{ horse-power.}$$

15. To find the time of oscillation of a circular pendulum.

Let P, P_1, P_2, P_3, P_4 fig. 12, be a material point attached to a thread or rod without weight, and oscillating in the plane of the

paper, about a fixed axis at X , the other extremity of the rod.

Taking O , the origin of the vertical and horizontal axes at the lowest point of the curve, put $r = OX$, $h = Oc$, $x = Oa$, $dx = ab$, $s = OP_3$, $ds = P_3e$, and since the object is to find the time it will take the body to move from the lowest point,

O to $P_4 \dots$ or, which

is the same thing, to move from P to the lowest point O —either of these branches might be investigated—the ascending one, $P_2, P_3, \dots$ is selected, then the arc s increases with the time, and $|\bar{s}$ which represents ds , in the well-known formula (A), is positive.

$$\left. \begin{array}{l} t, \text{ time employed in describing} \\ \text{the arc } PP_2 \text{ or } P_2P_4 \end{array} \right\} = \int \frac{|\bar{s}}{(2g(h-x))^{\frac{1}{2}}} \dots \dots \dots (A).$$

It must also be borne in mind, that in plane curves referred to rectangular co-ordinates

$$|\bar{s}| = \left(1 + \frac{|\bar{y}^2|}{|\bar{x}^2|}\right)^{\frac{1}{2}} |\bar{x}| \dots \dots \dots (B).$$

The equation of the path of P is $y^2 = 2rx - x^2$, putting $y = aP_1 = aP_3$.

$$\therefore \frac{|\bar{y}^2|}{|\bar{x}^2|} = \frac{(r-x)^2}{2rx - x^2}, \text{ and } |\bar{s}| = \frac{r|\bar{x}|}{(2rx - x^2)}.$$

$$\begin{aligned} \therefore t &= \int \frac{|\bar{s}|}{\{2g(h-x)\}^{\frac{1}{2}}} = \frac{r}{(2g)^{\frac{1}{2}}} \int \frac{|\bar{x}|}{\{(h-x)(2rx-x^2)\}^{\frac{1}{2}}} \\ &= \frac{(r)^{\frac{1}{2}}}{2(g)^{\frac{1}{2}}} \int_0^h \frac{|\bar{x}|}{(hx-x^2)} \left(1 - \frac{x}{2r}\right)^{-\frac{1}{2}}; \end{aligned}$$

if the second factor be developed by the binomial theorem, the differential in question will be reduced to a series of others of the known integral form

$$\frac{x^m |\bar{x}|}{(hx-x^2)^{\frac{1}{2}}}.$$

Consequently, the value of $2t$, or the time of a complete oscillation,

$$\begin{aligned} &= \pi \left(\frac{r}{g}\right)^{\frac{1}{2}} \left\{ 1 + \left(\frac{1}{2}\right)^2 \frac{h}{2r} + \left(\frac{1.3}{2.4}\right)^2 \left(\frac{h}{2r}\right)^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 \left(\frac{h}{2r}\right)^3 \right. \\ &\quad \left. + \left(\frac{1.3.5.7}{2.4.6.8}\right)^2 \left(\frac{h}{2r}\right)^4 + \dots \right\}. \end{aligned}$$

In this expression h is the versine of the arc of half the whole path $P \dots P_4$, and $\frac{h}{r}$ is the versine of a similar arc to radius 1. Let the arc $POP_4 = 29^\circ 14'$, for the sake of example, then $OP_4 = OP = 14^\circ 37'$, the natural versine of $14^\circ 37' = .0323642 = \frac{h}{r}$.

Suppose $r = 4.35$ feet, then $h = .1407843$ feet = 1.6894116 inches.

When $g = 32.18$ feet, it is required to find the time of one oscillation.

$$\frac{h}{2r} = .0161821 = .016 \downarrow 0, 1, 1, 3, 5,$$

$(.016)^4$ does not give a unit in the seventh decimal place.

$$(.016)^3 = .0000041$$

$$\left(\frac{1.3.5}{2.4.6}\right)^2 = \left(\frac{5}{16}\right)^2 = \frac{25}{256}$$

$$41 \times \frac{25}{256} = 4$$

$$\downarrow 0, 3, 3, 9, 15,$$

$$\begin{array}{r|l} 4 & \dots \\ \cdot & \dots \\ \hline & 4 \end{array}$$

So that $\left(\frac{1.3.5}{2.4.6}\right)^2 \left(\frac{h}{2r}\right)^3$ only gives .0000004.

$$(\cdot 016)^2 = \cdot 0002560$$

$$\left(\frac{1.3}{2.4}\right)^2 = \frac{9}{64}, \text{ and } \frac{9}{64} \times 2560 = 360.$$

$$\downarrow 0, 2, 2, 6, 10,$$

$$\begin{array}{r|l} 36 & 0 \dots \\ 7 & \dots \\ \hline 367 & \dots \\ 1 & \dots \\ \hline 368 & \end{array}$$

$$\therefore \left(\frac{1.3}{2.4}\right)^2 \left(\frac{h}{2r}\right)^2 = \cdot 0000368$$

$$\left(\frac{1}{2}\right)^2 \frac{h}{2r} = \cdot 0040455$$

$$\begin{array}{r} 1 \cdot 0000000 \\ \cdot 0040455 \\ 368 \\ \hline 4 \end{array}$$

$\downarrow 0, 0, 4, 0, 7, 6, 4, = 1 \cdot 0040827 =$ the sum of the series,
exact to seven places of decimals.

$$g = 32 \cdot 18 = (5 \cdot 6)^2 \downarrow 0, \quad 2, \quad 6, \quad 0, \quad \bar{8}, \quad \bar{4}, \quad \bar{16},$$

$$r = 4 \cdot 35 = (2)^2 \downarrow 0, \quad 8, \quad 4, \quad 2, \quad 8, \quad 0, \quad 8,$$

$$\pi = 3 \cdot 12 \downarrow 0, \quad 0, \quad 6, \quad 9, \quad 0, \quad 0, \quad \bar{2},$$

$$\pi \left(\frac{r}{g}\right)^{\frac{1}{2}} \times \text{the sum of series} = \frac{3 \cdot 12 \times 2}{5 \cdot 6} \downarrow 0, 3, 9, 10, 15, 8, 14,$$

$$= \frac{7 \cdot 8}{7} \downarrow 0, 4, 0, 2, 0, 4, 5,$$

$$\begin{aligned} \text{because } & \downarrow 0, \quad 3, \quad 9, \quad 10, \quad 15, \quad 8, \quad 14, \\ & = \downarrow 0, \quad 0, \quad 39, \quad 10, \quad 3, \quad \overline{4}, \quad \overline{7}, \\ & = \downarrow 0, \quad 0, \quad 0, \quad 400, \quad 3, \quad \overline{4}, \quad \overline{163}, \\ & = \downarrow 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 400097, \end{aligned}$$

Then by the method shown in Example 9 (page 131), *

$$\begin{aligned} & = \downarrow 0, \quad 4, \quad 0, \quad 0, \quad 0, \quad 0, \quad 2045, \\ & = \downarrow 0, \quad 4, \quad 0, \quad 2, \quad 0, \quad 4, \quad 5, \end{aligned}$$

$\frac{7 \cdot 8}{7} \downarrow 0, 4, 0, 2, 0, 4, 5, = 1 \cdot 1597672$ seconds, the time of one oscillation.

16. *By the direct application of Dual Arithmetic, and without the use of the well-known constant π , it is required to find the length of an arc to radius 1, corresponding to any number of degrees, minutes, and seconds, true to seven places of decimals, and give an example. (see page 86.)*

Let the arc whose length is required contain $12^\circ 15' 18'' \cdot 2$
 $= 44118'' \cdot 2$.

RULE:—The seconds multiplied by 4 and divided by 1000000, the result multiplied by $\downarrow 2, 0, 2, \overline{3}, \overline{2}, 0, 7$, gives the length of the given arc.

$$\begin{array}{r} 44118 \cdot 2 \\ \quad 4 \\ \hline \cdot 1 \overline{) 764728} \text{ seconds} \times 4 \text{ and divided by } 1000000 \\ \quad \underline{352945} \\ \quad \quad \underline{17647} \\ 213 \overline{) 5320} \dots \\ \quad \underline{4271} \dots \\ \quad \quad \underline{2} \dots \\ \hline 2139593 \cdot \end{array}$$

$$\begin{array}{r}
 2139 \overline{) 593} \cdot \\
 \underline{642} \quad \dots \quad \text{minus} \\
 43 \overline{) \dots} \quad \text{minus} \\
 \underline{1} \quad \dots \quad \text{plus} \\
 \hline
 2138909 \text{ length of arc.}
 \end{array}$$

17. *Having given the length of an arc to radius 1, to find the degrees, minutes, &c. contained in it, without the direct use of the constant π , by Dual Arithmetic, and give an example.*

How many degrees, minutes, &c. are contained in an arc to radius 1, whose length = $\cdot 2138909$?

RULE:—Multiply the length of the given arc by 200000, and then by $\downarrow 0,3,1,0,0,\overline{7},0,\overline{5}$, the result gives the number of seconds contained in the arc.

$$\begin{array}{r}
 \cdot 2138909 \\
 \quad \quad \quad 2 \\
 \hline
 42 \overline{) 778 \cdot 18} \cdot \text{arc multiplied by 200000.} \\
 \quad 1 \overline{) 28335} \cdot \\
 \quad \quad 1 \overline{) 283} \cdot \\
 \quad \quad \quad 4 \cdot \\
 \hline
 440 \overline{) 7440} \cdot \cdot \\
 \quad \quad 440 \overline{) 7} \cdot \cdot \\
 \hline
 441184 \overline{) 7} \cdot \cdot \cdot \cdot \cdot \\
 \quad \quad \quad 3 \overline{) 1} \cdot \cdot \cdot \cdot \cdot \text{minus}
 \end{array}$$

seconds $44118 \cdot 16 = 12^\circ 15' 18'' \cdot 16$.

18. *Suppose the apparent distance between the centres of the sun and moon to be $56^\circ 56' 31''$ (d), the apparent altitude of the moon's centre, $23^\circ 3' 4''$ (a), the apparent altitude of the sun's centre, $58^\circ 4' 35''$ (a), the true altitude of the moon's centre, $23^\circ 51' 42''$ (A), and the true altitude of the sun's centre, $58^\circ 3' 59''$ (A); find the true distance (D), so as to determine the longitude at sea.*

$$\cos D = [\cos d + \cos (a + a_)] \frac{\cos A \cos A_}{\cos a \cos a_} - \cos (A + A_).$$

$$\begin{array}{r}
 (d) 56^\circ 56' 31'' \cos = \cdot 5454885 \\
 (a + a_) 81 \quad 7 \quad 39 \cos = \cdot 1542362 \\
 \hline
 \cdot 6997247
 \end{array}$$

$$(A) 58^{\circ} 3' 59'' \cos = \cdot 5289\ 3\ 63$$

$$(a) 58\ 4\ 35 \cos = \cdot 5287\ 8\ 82 \cdot \downarrow 0,0,0,3,2,0,0,$$

$$1\ 5\ 86 \cdot$$

$$\begin{array}{r} 5289\ 4\ 68 \dots \\ 1\ 06 \dots \text{minus} \end{array}$$

$$5289\ 3\ 62$$

$$(a,) 23^{\circ} 3' 4'' \cos = \cdot 920\ 1\ 5\ 5\ 9 \quad \downarrow 0,0,6,1,4,1,7,$$

$$(A,) 23\ 51\ 42 \cos = \cdot 914\ 5\ 2\ 4\ 8 \dots$$

$$5\ 4\ 8\ 7\ 1 \dots$$

$$1\ 3\ 7 \dots$$

$$\begin{array}{r} \cdot 920\ 0\ 2\ 5\ 6 \cdot \\ 9\ 2\ 0 \cdot \end{array}$$

$$\begin{array}{r} \cdot 920\ 1\ 1\ 7\ 6 \dots \\ 3\ 6\ 8 \dots \end{array}$$

$$\begin{array}{r} \cdot 920\ 1\ 5\ 4\ 4 \dots\dots\dots \\ 9 \dots\dots\dots \end{array}$$

$$\begin{array}{r} \cdot 920\ 1\ 5\ 5\ 3 \dots\dots\dots \\ 6 \dots\dots\dots \end{array}$$

$$\cdot 920\ 1\ 5\ 5\ 9$$

$$\frac{\downarrow 0,0,0,3,2,0,0,}{\downarrow 0,0,6,1,4,1,7,} = \downarrow 0,0,6,2,6,1,7,$$

$$\begin{array}{r} 699\ 7\ 2\ 4\ 7 \dots + \\ 4\ 1\ 9\ 8\ 3 \dots - \\ 1\ 4\ 7 \dots + \end{array}$$

$$\begin{array}{r} 695\ 5\ 4\ 1\ 1 \cdot \\ 1\ 3\ 9\ 1 \cdot \end{array}$$

$$\begin{array}{r} 695\ 6\ 8\ 0\ 2 \dots + \\ 4\ 1\ 7 \dots\dots - \\ 7 \dots\dots\dots - \\ 5 \dots\dots\dots - \end{array}$$

$$\cdot 695\ 6\ 3\ 7\ 3$$

$$(A + A,) 81^{\circ} 55' 41'' \cos = \cdot 140\ 4\ 1\ 6\ 4$$

$$(D) 56\ 16\ 26 \cdot 6 \cos = \cdot 555\ 2\ 2\ 0\ 9$$

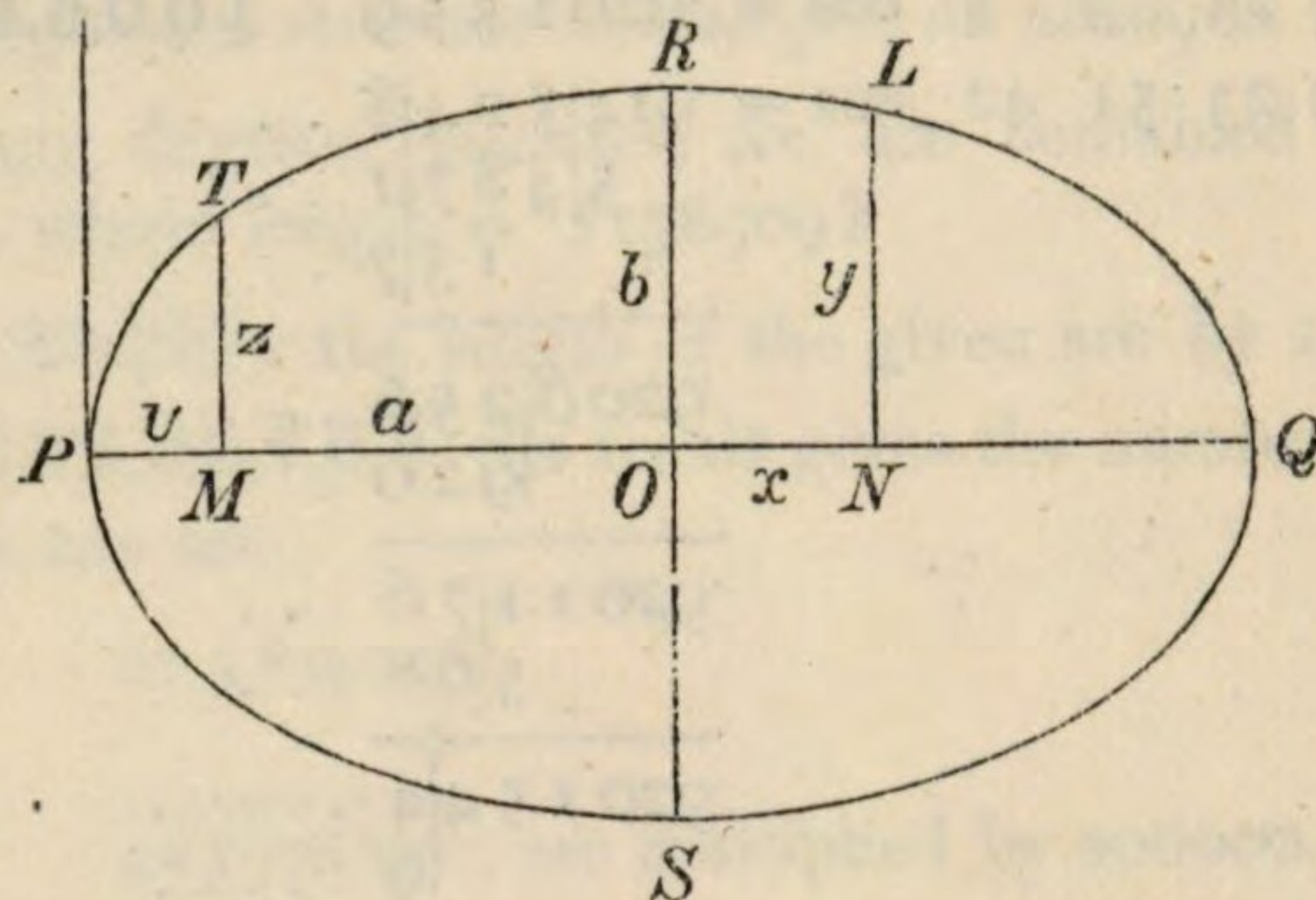
19. To find the length of an arc of an ellipsis.

Let $a = OP$, the semi-transverse axis,

$b = OR$, the semi-conjugate axis,

$x = ON$, and $y = NL$. s = the length of the arc RL .

Fig. 14.



Then, $a^2 : b^2 :: a^2 - x^2 : y^2 = \frac{b^2}{a^2} (a^2 - x^2)$, $y = \frac{b}{a} (a^2 - x^2)^{\frac{1}{2}}$.

$$(1) \quad \therefore a^2 y^2 + b^2 x^2 = a^2 b^2$$

In plane curves referred to rectangular co-ordinates,

$$(2) \quad s = \int \left(1 + \frac{|y^2|}{|x^2|} \right)^{\frac{1}{2}} |x|.$$

The differential of (1) gives $a^2 y \bar{y} + b^2 x \bar{x} = 0$,

$$\therefore \frac{|y^2|}{|x^2|} = \frac{b^4 x^2}{a^4 y^2} = \frac{b^2 x^2}{a^4 - a^2 x^2}$$

$\therefore s = \int \left(1 + \frac{b^2 x^2}{a^4 - a^2 x^2} \right)^{\frac{1}{2}} |x|$, which, by converting $\frac{b^2 x^2}{a^4 - a^2 x^2}$ into an infinite series, becomes

$$(3) \quad s = \int \left(1 + \frac{b^2}{a^4} x^2 + \frac{b^2}{a^6} x^4 + \frac{b^2}{a^8} x^6 + \frac{b^2}{a^{10}} x^8 + \dots \right)^{\frac{1}{2}} |x|.$$

Let the square root of the quantity enclosed in the brackets of (2) be represented by

$$(4) \ 1 + Ax^2 + Bx^4 + Cx^6 + Dx^8 + Ex^{10} + Fx^{12} + Gx^{14} + \dots$$

The square of (4) will be represented by the sum of the following series :

$$1 + 2Ax^2 + 2Bx^4 + 2Cx^6 + 2Dx^8 + 2Ex^{10} + 2Fx^{12} + 2Gx^{14} + \dots$$

$$A^2x^4 + 2ABx^6 + 2ACx^8 + 2ADx^{10} + 2AEx^{12} + 2AFx^{14} + \dots$$

$$B^2x^8 + 2BCx^{10} + 2BDx^{12} + 2BEx^{14} + \dots$$

$$C^2x^{12} + 2CDx^{14} + \dots$$

$$\therefore 2A = \frac{b^2}{a^4} \quad A = \frac{b^2}{2a^4} \quad \therefore \int \frac{b^2}{2a^4} x^2 | \bar{x} = \frac{b^2}{a^2} \frac{x^2}{a^2} \frac{x}{6}$$

$$A^2 + 2B = \frac{b^2}{a^6};$$

$$2B = \frac{b^2}{a^6} - \frac{b^4}{4a^8}; \quad B = \frac{b^2}{2a^6} - \frac{b^4}{8a^8};$$

$$\int \left(\frac{b^2}{a^2} - \frac{b^4}{4a^4} \right) \frac{x^4 | \bar{x}}{2a^4} = \left(\frac{b^2}{a^2} - \frac{b^4}{4a^4} \right) \frac{x^4}{a^4} \frac{x}{10}.$$

$$2AB + 2C = \frac{b^2}{a^8};$$

$$2C = \frac{b^2}{a^8} - \frac{b^4}{2a^{10}} + \frac{b^6}{8a^{12}}; \quad C = \frac{b^2}{2a^8} - \frac{b^4}{4a^{10}} + \frac{b^6}{16a^{12}};$$

$$\int \left(\frac{b^2}{a^2} - \frac{b^4}{2a^4} + \frac{b^6}{8a^6} \right) \frac{x^6 | \bar{x}}{2a^6} = \left(\frac{b^2}{a^2} - \frac{b^4}{2a^4} + \frac{b^6}{8a^6} \right) \frac{x^6}{a^6} \frac{x}{14}.$$

$$B^2 + 2AC + D = \frac{b^2}{a^{10}};$$

$$D = \frac{b^2}{2a^{10}} - \frac{3b^4}{8a^{12}} + \frac{3b^6}{16a^{14}} - \frac{5b^8}{128a^{16}};$$

$$\int \left(\frac{b^2}{a^2} - \frac{3b^4}{4a^4} + \frac{3b^6}{8a^6} - \frac{5b^8}{64a^8} \right) \frac{x^8 | \bar{x}}{2a^8} = \left(\frac{b^2}{a^2} - \frac{3b^4}{4a^4} + \frac{3b^6}{8a^6} - \frac{5b^8}{64a^8} \right) \frac{x^8}{a^8} \frac{x}{18}.$$

&c. = &c.

$$\begin{aligned}
 \therefore s &= x + \frac{b^2 x^2 x}{a^2 a^2 6} \\
 &+ \left(\frac{b^2}{a^2} - \frac{b^4}{4a^4} \right) \frac{x^4 x}{a^4 10} + \left(\frac{b^2}{a^2} - \frac{b^4}{2a^4} + \frac{b^6}{8a^6} \right) \frac{x^6 x}{a^6 14} \\
 &+ \left(\frac{b^2}{a^2} - \frac{3b^4}{4a^4} + \frac{3b^6}{8a^6} - \frac{5b^8}{64a^8} \right) \frac{x^8 x}{a^8 18} + \dots
 \end{aligned}$$

When the length of the arc RL is found, the length of LQ = the quadrant RQ minus RL .

Numerical Example.

Let $a = 41.3245$; $b = 25.1636$; $x = 12.3456$; $x = 10 \downarrow 2, 2, 0, 2, 0, \overline{6}$,

$$\frac{b}{a} = .6089269 = .6 \downarrow 0, 1, 4, 8, 2, 0, 3,$$

$$\frac{x}{a} = .2987477 = .3 \downarrow 0, 0, \overline{4}, \overline{2}, 1, 5,$$

$$\begin{aligned}
 \frac{b^2}{a^2} &= .36 \downarrow 0, 2, 8, 16, 4, 0, 6, \\
 &= .36 \downarrow 0, 2, 9, 6, 4, 1, 0, \\
 &= .3707918
 \end{aligned}$$

$$\begin{aligned}
 \frac{b^4}{a^4} &= .1296 \downarrow 0, 4, 16, 32, 8, 0, 12, \\
 &= .1296 \downarrow 0, 5, 9, 3, 2, 7, 1, \\
 &= .1374867
 \end{aligned}$$

$$\begin{aligned}
 \frac{b^6}{a^6} &= .046656 \downarrow 0, 6, 24, 48, 12, 0, 18, \\
 &= .046656 \downarrow 0, 8, 9, 0, 0, 3, 2, \\
 &= .0509784
 \end{aligned}$$

$$\begin{aligned}
 \frac{b^8}{a^8} &= .0167962 \downarrow 0, 8, 32, 64, 16, 0, 24, \\
 &= .0167962 \downarrow 1, 1, 2, 9, 3, 8, 7, \\
 &= .0187155
 \end{aligned}$$

$$\frac{b^2}{a^2} = .3707918 = k$$

$$\frac{b^2}{a^2} - \frac{b^4}{4a^4} = .3364201 = k \downarrow \bar{1}, 0, \bar{2}, 0, 3, \bar{1}, = l$$

$$\frac{b^2}{a^2} - \frac{b^4}{2a^4} + \frac{b^6}{8a^6} = .3084207 = l \downarrow 0, \bar{8}, \bar{7}, \bar{3}, 0, 3, = m$$

$$\frac{b^2}{a^2} - \frac{3b^4}{4a^4} + \frac{3b^6}{8a^6} - \frac{5b^8}{64a^8} = .2867937 = m \downarrow 0, \bar{7}, \bar{3}, 0, \bar{5}, \bar{1}, = n$$

From analogy it becomes apparent that the next terms are nearly

}	$= n \downarrow 0, \bar{6},$	$= p$
	$= p \downarrow 0, \bar{5},$	$= q$

$$\frac{b^2}{a^2} = .36 \downarrow 0, 2, 9, 6, 4, 1,$$

$$\frac{x^2}{a^2} = .09 \downarrow 0, 0, \bar{8}, \bar{4}, 3, 0,$$

$$x = 10 \cdot \downarrow 2, 2, 0, 2, 0, \bar{6},$$

$$.324 \downarrow 2, 4, 1, 4, 7, \bar{5}, = k \frac{x^2}{a^2} x$$

$$\therefore \frac{b^2}{a^2} \frac{x^2}{a^2} \frac{x}{6} = .054 \downarrow 2, 4, 1, 4, 7, \bar{5}, = .0680928.$$

$$k \frac{x^2}{a^2} x = .324 \downarrow 2, 4, 1, 4, 7, \bar{5},$$

$$\frac{x^2}{a^2} = .09 \downarrow 0, 0, \bar{8}, \bar{4}, 3, 0,$$

$$\downarrow \bar{1}, 0, \bar{2}, 0, 3, \bar{1},$$

$$l \frac{x^4}{a^4} x = .02916 \downarrow 1, 4, \bar{9}, 0, 13, \bar{6},$$

$$\left(\frac{b^2}{a^2} - \frac{b^4}{4a^4} \right) \frac{x^4}{a^4} \frac{x}{10} = .002916 \downarrow 1, 4, \bar{9}, 1, 3, \bar{6}, = .0033081$$

$$l \frac{x^4}{a^4} x = \cdot 02916 \downarrow 1, 4, \bar{9}, 0, 13, \bar{6},$$

$$\frac{x^2}{a^2} = \cdot 09 \downarrow 0, 0, \bar{8}, \bar{4}, 3, 0,$$

$$\downarrow 0, \bar{8}, \bar{7}, \bar{3}, 0, 3,$$

$$m \frac{x^6}{a^6} x = \cdot 0026244 \downarrow 1, \bar{4}, \bar{24}, \bar{7}, 16, \bar{3},$$

$$\left(\frac{b^2}{a^2} - \frac{b^4}{2a^4} + \frac{b^6}{8a^6} \right) \frac{x^6}{a^6} \frac{x}{14} = \cdot 0001875 \downarrow 0, 3, 1, 1, 2, 7, = \cdot 0001933$$

$$m \frac{x^6}{a^6} x = \cdot 0026244 \downarrow 0, 3, 1, 1, 2, 7,$$

$$\frac{x^2}{a^2} = \cdot 09 \downarrow 0, 0, \bar{8}, \bar{4}, 3, 0,$$

$$\downarrow 0, \bar{7}, \bar{3}, 0, \bar{5}, \bar{1},$$

$$n \frac{x^8}{a^8} x = \cdot 0002362 \downarrow 0, \bar{4}, \bar{10}, \bar{3}, 0, 6,$$

$$\left(\frac{b^2}{a^2} - \frac{3b^4}{4a^4} + \frac{3b^6}{8a^6} - \frac{5b^8}{64a^8} \right) \frac{x^8}{a^8} \frac{x}{18} = \cdot 0000131 \downarrow 0, \bar{4}, \bar{10}, \bar{3}, 0, 6,$$

$$= \cdot 0000124.$$

$$n \frac{x^8}{a^8} x = \cdot 0002362 \downarrow 0, \bar{4}, \bar{10}, \bar{3}, 0, 6,$$

$$\frac{x^2}{a^2} = \cdot 09 \downarrow 0, 0, \bar{8}, \bar{4}, 3, 0,$$

$$\downarrow 0, \bar{6},$$

$$p \frac{x^{10}}{a^{10}} x = \cdot 0000213 \downarrow 0, \bar{10}, \bar{18},$$

$$p \frac{x^{10}}{a^{10}} \frac{x}{22} = \cdot 0000009 \downarrow \bar{1}, \bar{1}, = \cdot 0000008$$

$$\begin{array}{r}
 x = 12.3456000 \\
 .0680928 \\
 .0033081 \\
 .0001933 \\
 .0000124 \\
 .0000008 \\
 \hline
 \end{array}$$

12.4172074 = length of elliptic arc RL .

To render the mode of operating clear, the work of this example is given in detail. If the axes of co-ordinates be at P , the extremity of the major axis, then, putting $PM = v$ and $MT = z$, the equation to the ellipsis will be $z^2 = \frac{b^2}{a^2} (2av - v^2)$; and the length of the arc PT may be found in a similar manner in terms of a , b , and the ordinate z . The length of an elliptic arc measured from P , is often required.

20. *To determine the numerical value of elliptic and hyperbolic functions.*

Suppose the expression (1) has to be integrated,

$$\frac{AX^2 \sqrt{X}}{(BX^2 - X^4 - C)^{\frac{1}{2}}} \dots \dots (1)$$

Take (2) the equation to the ellipsis,

$$a^2 y^2 + b^2 x^2 = a^2 b^2 \dots \dots (2)$$

the differential of the arc of the ellipse, by (2) of the last problem, is

$$\sqrt{s} = \frac{[a^4 - (a^2 - b^2) x^2]^{\frac{1}{2}} \sqrt{x}}{a (a^2 - x^2)^{\frac{1}{2}}} \dots \dots (3)$$

$$\text{Assume } [a^4 - (a^2 - b^2) x^2]^{\frac{1}{2}} = av \dots \dots (4)$$

solve (4) for x , then

$$x = \frac{a (a^2 - v^2)^{\frac{1}{2}}}{(a^2 - b^2)^{\frac{1}{2}}} \dots \dots (5)$$

Put the values of x and $\sqrt{x}$ from (5) into (3), then

$$\sqrt{s} = \frac{v^2 \sqrt{v}}{[(a^2 + b^2) v^2 - v^4 - a^2 b^2]^{\frac{1}{2}}} \dots \dots (6)$$

(6) is the same form as (1). It is evident that the integral of (1) is the arc of an ellipse. To find the axes of the ellipse, equate the co-efficients of the like powers of x and v under the radicals of (6) and (1); that is, put

$$a^2 + b^2 = B \text{ and } a^2 b^2 = C,$$

from which the axes a and b may be found. The abscissa x , of the extremity of the elliptic arc (1) is given by (5). Again, suppose the integral of (7) is required:

$$\frac{A X}{X^2 (B X^2 - X^4 - C)^{\frac{1}{2}}} \dots \dots (7).$$

Assume the numerator of (3) = $\frac{a^2 b}{v}$, that is,

$$[a^4 - (a^2 - b^2) x^2]^{\frac{1}{2}} = \frac{a^2 b}{v} \dots \dots (8).$$

From (8), (9) is readily obtained:

$$x = \frac{a^2 (v^2 - b^2)^{\frac{1}{2}}}{v (a^2 - b^2)^{\frac{1}{2}}} \dots \dots (9).$$

This value of x , and its differential, substituted in (3), gives

$$\sqrt{s} = \frac{-a^2 b^2 \sqrt{v}}{v^2 [(a^2 + b^2) v^2 - v^4 - a^2 b^2]^{\frac{1}{2}}} \dots \dots (10)$$

Hence, the integral of (7) is an elliptic arc, the abscissa of whose extremity is the value of x in (9); the axes may be found as in the last case.

Take another example, and suppose the integral of (11) to be required:

$$\frac{A X^2 \sqrt{X}}{(B X^2 + X^4 - C)^{\frac{1}{2}}} \dots \dots (11).$$

Take (12) as the equation of the hyperbola,

$$a y^2 - b^2 x^2 = - a^2 b^2 \dots\dots\dots (12)$$

Then

$$l\bar{s} = \frac{[(a^2 + b^2) x^2 - a^4]^{\frac{1}{2}} l\bar{x}}{a^2 (x^2 - a^2)^{\frac{1}{2}}} \dots\dots\dots (13)$$

$$\text{Assume } (a^2 + b^2) x^2 - a^4 = a^2 z^4 \dots\dots\dots (14),$$

then (13) becomes, in terms of z , reduced to (15):

$$l\bar{s} = \frac{z^2 l\bar{z}}{[(a^4 - b^2) z^2 + z^4 - a^2 b^2]^{\frac{1}{2}}} \dots\dots\dots (15).$$

(15) is of the same form as (11), $\therefore$ (11) is the differential of an arc of an hyperbola, whose axes and abscissa are found as in the case of the elliptic arc.

Lastly, suppose the form (16) has to be integrated :

$$\frac{A l\bar{X}}{X^2 (B X^2 - X^4 + C)^{\frac{1}{2}}} \dots\dots\dots (16).$$

By assuming $(a^2 + b^2) x^2 - a^4 = \frac{a^2 b^2}{z^2} \dots\dots\dots (17)$, then $l\bar{s}$ becomes of the same form as (16), which is therefore the differential of an arc of an hyperbola.

Example.

21. *Required the integral of (18), when $x = 28$, neglecting the constant.*

$$\frac{200 x^2 l\bar{x}}{(1000 x^2 - x^4 - 130000)^{\frac{1}{2}}} \dots\dots\dots (18).$$

$$a^2 + b^2 = 1000, \text{ and } a^2 b^2 = 130000, \text{ by 6.}$$

$$\therefore (a^2 - b^2)^2 = 480000,$$

$$a^2 - b^2 = 692.82032.$$

$$a^2 = 846.41016, a = 29.093129; b^2 = 153.58984, b = 12.393137.$$

The integral of (18) is 200 times an arc of an ellipse whose semi-transverse axis = 29.093129, and semi-conjugate = 12.393137, and the square of the abscissa (5) of the extremity of this elliptic arc equals

$$\frac{a^2 (a^2 - x^2)}{(a^2 - b^2)}, \text{ which put } = Z^2.$$

$$a^2 = 29^2 \downarrow 0,0,6,4,0,16,0, \quad a^2 - x^2 = (7.9)^2 \downarrow 0,0,0,0,0,2,6,$$

$$a^2 - b^2 = (26)^2 \downarrow 0,2,4,6,8,0,\overline{10},$$

$$\therefore Z = \frac{29 \times 7.9}{26} \downarrow 0,\overline{1},\overline{1},\overline{1},\overline{4},9,8, = 8.8115385 \downarrow 0,\overline{1},\overline{1},\overline{1},\overline{4},9,8,$$

$$Z = 8.7318828 = 8. \downarrow 0,9,\overline{2},0,\overline{2},5,5,$$

$$\frac{Z}{a} = \left(\frac{a^2 - x^2}{a^2 - b^2} \right)^{\frac{1}{2}} = \frac{7.9}{26} \downarrow 0,\overline{1},\overline{2},\overline{3},\overline{4},1,8 = .3001354 = .3 \downarrow 0,0,0,4,5,1,$$

$$\frac{Z^2}{a^2} = .09 \downarrow 0,0,0,9,0,2,$$

$$\frac{b^2}{a^2} = .1814603 = .18 \downarrow 0,0,8,0,8,0,4,$$

$$\text{Length of arc} = Z + \frac{b^2}{a^2} \frac{z^2}{a^2} \frac{Z}{6} + \left(\frac{b^2}{a^2} - \frac{b^4}{4a^4} \right) \frac{z^4}{a^4} \frac{Z}{10}$$

$$+ \left(\frac{b^2}{a^2} - \frac{b^4}{2a^4} + \frac{b^6}{8a^6} \right) \frac{z^6}{a^6} \frac{Z}{14}$$

$$+ \left(\frac{b^2}{a^2} - \frac{3b^4}{4a^4} + \frac{3b^6}{8a^6} - \frac{5b^8}{64a^8} \right) \frac{z^8}{a^8} \frac{Z}{18} + \dots$$

$$\frac{b^2}{a^2} = .18 \downarrow 0,0,8,0,8,0,4,$$

$$= .1814603$$

$$\begin{aligned}\frac{b^4}{a^4} &= \cdot 0324 \downarrow 0, 0, 16, 0, 16, 0, 8, \\ &= \cdot 0324 \downarrow 0, 1, 6, 2, 0, 5, 5, \\ &= \cdot 0329271\end{aligned}$$

$$\begin{aligned}\frac{b^6}{a^6} &= \cdot 005832 \downarrow 0, 0, 24, 0, 24, 0, 12, \\ &= \cdot 005832 \downarrow 0, 2, 4, 1, 1, 4, 6, \\ &= \cdot 0059737\end{aligned}$$

$$\begin{aligned}\frac{b^8}{a^8} &= \cdot 0010498 \downarrow 0, 0, 32, 0, 32, 0, 16, \\ &= \cdot 0010498 \downarrow 0, 3, 2, 4, 5, 6, 3, \\ &= \cdot 0010843\end{aligned}$$

$$\frac{b^2}{a^2} = \cdot 1814603 = k.$$

$$\frac{b^2}{a^2} - \frac{b^4}{4a^4} = \cdot 1732285 = k \downarrow 0, \bar{5}, 3, 3, 2, 7, = l.$$

$$\frac{b^2}{a^2} - \frac{b^4}{2a^4} + \frac{b^6}{8a^6} = \cdot 1657434 = l \downarrow 0, \bar{5}, 5, 6, 1, \bar{5}, = m.$$

$$\frac{b^2}{a^2} - \frac{3b^4}{4a^4} + \frac{3b^6}{8a^6} - \frac{5b^8}{64a^8} = \cdot 1590870 = m \downarrow 0, \bar{5}, 8, 7, 6, 7, = n.$$

$$\dots\dots\dots = n \downarrow 0, \bar{5}, 12, = p.$$

$$\dots\dots\dots = p \downarrow 0, \bar{5}, 16, = q.$$

$$\frac{b^2}{a^2} = \cdot 18 \downarrow 0, 0, 8, 0, 8, 0, 4,$$

$$\frac{z^2}{a^2} = \cdot 09 \downarrow 0, 0, 0, 9, 0, 2,$$

$$Z = 8 \cdot \downarrow 0, 9, \bar{2}, 0, \bar{2}, 5, 5,$$

$$\cdot 1296 \downarrow 0, 9, 6, 9, 6, 7, 9, = k \frac{z^2}{a^2} Z.$$

$$\therefore \frac{b^2}{a^2} \frac{Z^2}{a^2} \frac{Z}{6} = \cdot 0216 \downarrow 0, 9, 6, 9, 6, 7, 9, = \cdot 0233040.$$

$$k \frac{z^2}{a^2} Z = \cdot 1296 \downarrow 0, 9, 6, 9, 6, 7, 9,$$

$$\frac{z^2}{a^2} = \cdot 09 \downarrow 0, 0, 0, 9, 0, 2,$$

$$\downarrow 0, \bar{5}, 3, 3, 2, 7,$$

$$\cdot 011664 \downarrow 0, 4, 9, 21, 8, 16, 9, = l \frac{z^4}{a^4} Z.$$

$$\therefore \left(\frac{b^2}{a^2} - \frac{b^4}{4a^4} \right) \frac{Z^4}{a^4} \frac{Z}{10} = \cdot 0011664 \downarrow 0, 5, 1, 2, 4, 2, 4, = \cdot 0012273.$$

$$l \frac{z^4}{a^4} Z = \cdot 011664 \downarrow 0, 5, 1, 2, 4, 2, 4,$$

$$\frac{Z^2}{a^2} = \cdot 09 \downarrow 0, 0, 0, 9, 0, 2,$$

$$\downarrow 0, \bar{5}, 5, 6, 1, \bar{5},$$

$$\cdot 0010498 \downarrow 0, 0, 6, 17, 5, \bar{1}, 4, = m \frac{z^6}{a^6} Z.$$

$$\left(\frac{b^2}{a^2} - \frac{b^4}{2a^4} + \frac{b^6}{8a^6} \right) \frac{Z^6}{a^6} \frac{Z}{14} = \cdot 0000750 \downarrow 0, 0, 7, 7, 5, \bar{1}, 8, = \cdot 0000756.$$

$$m \frac{z^6}{a^6} Z = \cdot 0010498 \downarrow 0, 0, 7, 7, 5,$$

$$\frac{Z^2}{a^2} = \cdot 09 \downarrow 0, 0, 0, 9, 0,$$

$$\downarrow 0, \bar{5}, 8, 7, 6,$$

$$\cdot 0000945 \downarrow 0, \bar{5}, 15, 23, 11, = n \frac{z^8}{a^8} Z.$$

$$\left(\frac{b^2}{a^2} - \frac{3b^4}{4a^4} + \frac{3b^6}{8a^6} - \frac{5b^8}{64a^8} \right) \frac{Z^8}{a^8} \frac{Z}{18} = \cdot 0000053 \downarrow 0, \bar{4}, 7, = \cdot 0000051.$$

$$n \frac{z^8}{a^8} Z = \cdot 0000945 \downarrow 0, \bar{4}, 7,$$

$$\frac{z^2}{a^2} = \cdot 09 \downarrow 0, 0, 0,$$

$$\downarrow 0, \bar{5}, 12,$$

$$\cdot 0000085 \downarrow 0, \bar{9}, 19, = p \frac{z^{10}}{a^{10}} Z$$

$$p \frac{z^{10}}{a^{10}} \frac{Z}{22} = \cdot 0000004 \downarrow 0, \bar{8}, 9, = \cdot 0000004$$

$$Z = 8.7318828 \text{ abscissa}$$

$$\cdot 0233040$$

$$\cdot 0012273$$

$$\cdot 0000756$$

$$\cdot 0000051$$

$$\cdot 0000004$$

$$8.7564952 \text{ length of the elliptic arc.}$$

Consequently the integral of (18) when $x = 28$, becomes 1751.29904, the constant not being taken into account.

22. Find the true meridional parts corresponding to latitude $83^\circ 25' 24''$.

	90°	$0'$	$0''$
	83	25	24
2)	173	25	24
	86	42	42
	90	0	0
	3°	$17'$	$18''$

The length of an arc of $3^\circ 17' 18'' = \cdot 057392242$, the sine and cosine of this arc are readily calculated.

$$\sin 3^\circ 17' 18'' = \cdot 0573607 = \cos 86^\circ 42' 42''.$$

$$\cos 3^\circ 17' 18'' = \cdot 9983465 = \sin 86^\circ 42' 42''.$$

$$\cdot 9983465 \downarrow 0,0,1,6,5,5,5,3, = 1.$$

$$\therefore \cdot 9983465 = \frac{1}{\downarrow 0,0,1,6,5,5,5,3,}$$

$$\cdot 0573607 = \frac{5}{10^2} \downarrow 1,4,2,2,2,5,9,1,$$

$$\frac{\sin}{\cos} = \tan = \frac{10^2}{5} \frac{1}{\downarrow 1,4,3,8,7,10,14,4,} = \frac{20}{\downarrow 1,4,3,8,8,1,4,4,}.$$

9541497

3930332

289865

88144

230270081 ~ 10

69318201 ~ 2

299588282

13899838

285688444

$$\downarrow 1,4,3,8,8,1,4,4, = \downarrow 13899838,$$

Then, $285688444 \div 100005025 = 2.8567409$, the hyperbolic logarithm of the tangent, or

$$\begin{array}{r|l} 2.8568 & 84 | 44 \\ \text{half} & 1 | 42 | 84 \\ \text{half} & & 71 \end{array}$$

1 43 55 subtract

2.8567 40 89,

which has to be multiplied by

$$3437.74679 = \frac{10000}{3} \downarrow 0,3,1,0,0,\bar{7},0,\bar{4},$$

OPERATION UNABRIDGED.

$$3 \) \ 285 \ 67.40 \ 89$$

$$95 \ | \ 22 \ | \ 46 \ | \ 96$$

$$2 \ | \ 85 \ | \ 67 \ | \ 41$$

$$2 \ | \ 85 \ | \ 67$$

$$95$$

$$98 \ 1 \ | \ 100 \ | \ 99 \ .$$

$$98 \ 1 \ | \ 10 \ .$$

$$98 \ 2 \ 0 \ 82 \ | \ 09$$

$$6 \ | \ 87 \ | \ \text{minus} \sim 7$$

$$4 \ | \ \text{minus} \sim 4$$

Meridional parts 98 2 0.75 18 for lat. $83^\circ 25' 24''$.

23. Find the value of $\left(\frac{1.3.5.7.9.11.13}{2.4.6.8.10.12.14}\right)^2 x^7$, a term in the series that expresses the time of oscillation of a circular pendulum, when $x = .113574657$.

By adding the values of 1.3.5. and also the values of 2.4.6. reduced to the eight position, and then taking the differences, the results will be

$$\frac{1}{2} = -69318201; \quad \frac{1.3.5.7}{2.4.6.8} = -129674734;$$

$$\frac{1.3}{2.4} = -98087853; \quad \frac{1.3.5.7.9}{2.4.6.8.10} = -140211315;$$

$$\frac{1.3.5}{2.4.6} = -116320925; \quad \frac{1.3.5.7.9.11}{2.4.6.8.10.12} = -148912889;$$

$$\frac{1.3.5.7.9.11.13}{2.4.6.8.10.12.14} = -156324061.$$

$$x = .113574657 = \frac{1}{10} \downarrow 1,3,2,1,4,7,6,5, = \downarrow -217538660,$$

$$\begin{array}{r} -217538660 \\ \quad \quad \quad 7 \\ \hline -1522770620 \\ -156324061 \times 2 = -312648122 \\ \hline -1835418742 \\ 10^8 \sim + 1842160648 \\ \hline 6741906 \\ 5970498 \sim \downarrow 0,6, \\ \hline 771408 \\ 699685 \sim 7, \\ \hline 7,1,7,2,3, \end{array}$$

$$\downarrow 0,6,7,7,1,7,2,3, = 1.06974005$$

∴ The value of the term will be .0000000106974005, true to the last figure. In this case the range of the pendulum would be a circular arc of $78^\circ 46' 42''.8$.

24. Let $x = \cdot 113574657$, the cosine of $83^\circ 28' 42'' \cdot 8$, to find the sine, tangent, secant, versine, cotangent, cosecant, coversine, and their hyperbolic logarithms.

$$\text{ver.} = 1 - x = \cdot 886425343 = \frac{8}{10} \downarrow 1,0,7,2,7,8,5,2, = \downarrow - \overline{12056443},$$

$$\text{cov.} = 1 + x = 1 \cdot 113574657 = \downarrow 1,1,2,3,1,5,5,0, = \downarrow \overline{10758040},$$

$$\cos = x = \frac{1}{10} \downarrow 1,3,2,1,4,7,6,5, = \downarrow - \overline{217538660},$$

$$\begin{array}{r} -12056443 \\ +10758040 \\ \hline 2) -1298403 \end{array}$$

$$\sqrt{(1+x)(1-x)} = \sqrt{1-x^2} = \downarrow - \overline{649202}, = \frac{9}{10} \downarrow 1,0,3,5,6,0,1,7,$$

$$10 \sim +230270081$$

$$229620879$$

$$9 \sim 219733500$$

$$9887379$$

$$9531497 \sim \downarrow 1,$$

$$355882$$

$$299865 (0,0,3,$$

$$5,6,0,1,7,$$

$$\therefore \sin = \frac{9}{10} \downarrow 1,0,3,5,6,0,1,7, = \cdot 993529332.$$

$$\frac{-649202}{100005025} = -\cdot 00649169 = \text{hyp. log. sin.}$$

$$-21753 \overline{)8660}$$

$$\text{half } 1 \overline{)08} 77$$

$$\text{half } \quad \quad \overline{)54}$$

$$10931 \text{ subtract}$$

$$-217527729 \text{ hyp. log. cos.}$$

$$\begin{array}{r} - \quad 12056 \overline{) 443} \\ \quad \quad 60 \overline{) 3} \\ \quad \quad \quad 3 \\ \hline \quad \quad 606 \end{array}$$

$$- \cdot 12055 \, 83 \, 7 \text{ hyp. log. ver.}$$

$$\begin{array}{r} + \quad 10758 \overline{) 040} \\ \quad \quad 53 \overline{) 8} \\ \quad \quad \quad 3 \\ \hline \quad \quad 541 \end{array}$$

$$\cdot 10757 \, 49 \, 9 \text{ hyp. log. cov.}$$

$$x : \sqrt{1 - x^2} :: 1 : \tan = \frac{\sqrt{1 - x^2}}{x}$$

$$\begin{array}{l} - \quad 64 \, 92 \, 02 \text{ from} \\ - \quad 2 \, 1753 \, 86 \, 60 \text{ take} \end{array}$$

$$+ 2 \, 1688 \overline{) 94 \, 58} = 8 \downarrow 0,8,9,7,4,5,9,5, = 8 \cdot 74765512 = \text{tangent.}$$

$$\begin{array}{r} 1 \overline{) 08} \overline{) 44} \\ \quad \quad 54 \\ \hline 1 \, 08 \, 98 \end{array}$$

$$2 \cdot 1687 \, 85 \, 60 \text{ hyp. log. tan.}$$

$$\sqrt{1 - x^2} : x :: 1 : \frac{x}{\sqrt{1 - x^2}} = \cot.$$

$$\begin{array}{l} - \quad 2 \, 1753 \, 86 \, 60 \text{ from} \\ - \quad 64 \, 92 \, 02 \text{ take} \end{array}$$

$$- 2 \, 1688 \overline{) 94 \, 58} = \frac{2}{10} \downarrow 1,7,8,0,4,4,5,8, = \cdot 237773952 = \text{cotangent.}$$

$$\begin{array}{r} 1 \overline{) 08} \overline{) 44} \\ \quad \quad 54 \\ \hline 1 \, 08 \, 98 \end{array}$$

$$- 2 \cdot 1687 \, 85 \, 60 = \text{hyp. log. cot.}$$

$$x : 1 :: 1 : \frac{1}{x} = \text{secant.}$$

$$\begin{array}{r}
 0\ 0000\ 00\ 00\ \text{from} \\
 -\ 2\ 1753\ 86\ 60\ \text{take} \\
 \hline
 +\ 2\ 1753\ 86\ 60 = 8\ \downarrow\ 1,0,0,6,1,5,5,9, = 8.80541864 \\
 \begin{array}{r}
 1\ 08\ 77 \\
 \hline
 1\ 54 \\
 \hline
 1\ 09\ 21
 \end{array} \\
 \hline
 2.1752\ 77\ 39 = \text{hyp. log. secant.}
 \end{array}$$

$$\sqrt{1-x^2} : 1 :: 1 : \frac{1}{\sqrt{1-x^2}} = \text{cosecant.}$$

$$\begin{array}{r}
 00000000\ 0\ \text{from} \\
 -\ 64920\ 2\ \text{take} \\
 \hline
 +\ 64920\ 2 = \downarrow\ 0,0,6,4,9,4,7,2, = 1.00651282 \\
 \begin{array}{r}
 3\ 3 \\
 \hline
 0064916\ 9 = \text{hyp. log. cosecant.}
 \end{array}
 \end{array}$$

$$\frac{-649202^1}{230270081} = -.002819302$$

$$\begin{array}{r}
 10.0000000000 \\
 002819302 \\
 \hline
 9.997180698 = \text{common log. sine of the given}
 \end{array}$$

arc. The com. log. cos. &c. may be determined in a similar manner.

GENERAL SOLUTION

OF

ALGEBRAICAL EQUATIONS OF ALL DEGREES.

To multiply a number by any of the factors ↓5,; ↓0,5,; ↓0,0,5,; &c. is an operation so simple, that future results in such cases will only be exhibited, the work being omitted.

(A) 132690018825

(B) 133354797147 151

Thus the number (A), multiplied by ↓0,0,5, produces the number (B). According to the following monogram, which is easily remembered, 151 is made to show that there are two zeros before the operating figure 5, or, which is the same thing, the number (A) operated upon is divided into periods of three figures each.

MONOGRAM.

one	two	three
four	five	six
seven	eight	nine

WORK WITHOUT CONTRACTION.

(A) 132690018825 ~ 1

(a) 663450094 ~ 5

(b) 1326900 ~ 10

(c) 1327 ~ 10

(d) 1 ~ 5

(B) 133 354 797 147

(A) advanced a figure and divided by 2, gives (a), falling back a period to the right.

(b) is the same as (A), advanced a figure, and falling back two periods to the right.

(c) is the same as (A), advanced a figure, and falling back three periods to the right hand.

(d) is the same as (a), but for $\cdot 663 \dots$ neglected 1 is brought forward to the next period. The process becomes plain, when the nature of the multipliers 1, 5, 10, 10, 5, 1, are examined. This property of the factors $\downarrow 5$, $\downarrow 0,5$, $\downarrow 0,0,5$, &c. was pointed out before.

$$\begin{array}{r|l} 12 & 50 \mid 00 \mid 00 \mid 00 \mid 00 \mid 00 \\ 13 & 13 \mid 76 \mid 25 \mid 62 \mid 62 \mid 50 \\ & 13 \mid 13 \mid 76 \mid 25 \mid 62 \mid 63 \end{array} \quad \underline{51}$$

$$13 \ 26 \ 90 \ 01 \ 88 \ 25 \ 13 \quad \underline{11} = \downarrow 0,6,$$

Since the result at the end of each step is only required, the figures between the steps may be omitted in the general statement. Should the work require revision, the figures left out are easily made to reappear by a trifling calculation. The last example may stand thus:—

$$\begin{array}{r|l} 12 & 5 \cdot 0 \mid 0 \mid 0 \mid 00 \mid 00 \mid 00 \mid 00 \\ 13 & 1 \cdot 3 \mid 7 \mid 6 \mid 25 \mid 62 \mid 62 \mid 50 \\ & 13 \mid 2 \cdot 6 \mid 9 \mid 0 \mid 0 \mid 1 \mid 88 \mid 25 \mid 13 \end{array} \quad \begin{array}{l} \underline{51} \\ \underline{11} \end{array} = 125 \cdot \downarrow 0,6,$$

Examples.

1. Extract the root of the equation $3x^2 + 4x = 21$, to eight places of decimals.

By substituting 2 for x , the result will approach 21; by this method 2 may, or not, be the first figure of the required root.

$$\begin{array}{r} + \\ 12 \cdot 00000000 \end{array} \quad \begin{array}{r} + \\ 8 \cdot 00000000 \end{array}$$

$$\begin{array}{r} \text{twice } 12 = 24 \\ \text{once } 8 = 8 \\ \hline 32 \end{array}$$

$$\begin{array}{r} 12 \cdot 00000000 \\ 8 \cdot 00000000 \\ \hline 20 \cdot 00000000 \text{ from} \\ 21 \cdot 00000000 \text{ take} \\ \hline 32) 1 \cdot 00000000 \downarrow 0,3, \end{array}$$

$$\begin{array}{r} 12 \cdot 00 \cdot 00 \cdot 00 \cdot 00 \\ 12 \cdot 61 \cdot 21 \cdot 20 \cdot 60 \quad \underline{51} \\ 12 \cdot 73 \cdot 82 \cdot 41 \cdot 81 \quad \underline{11} \end{array}$$

$$\begin{array}{r} 80 \cdot 00 \cdot 00 \cdot 00 \cdot 0 \\ 82 \cdot 42 \cdot 40 \cdot 80 \cdot 0 \quad \underline{31} \end{array}$$

$$\begin{array}{r} \text{twice } 2546 \\ \text{once } 824 \\ \hline 3370 \end{array}$$

$$\begin{array}{r} 12 \cdot 73 \cdot 8241 \cdot 81 \\ 8 \cdot 24 \cdot 2408 \cdot 00 \\ \hline 20 \cdot 98 \cdot 0649 \cdot 81 \text{ take} \\ 21 \cdot 00 \cdot 0000 \cdot 00 \text{ from} \\ \hline 3370) 1 \cdot 9350 \cdot 19 (\downarrow 0,0,0,5, \end{array}$$

$$\begin{array}{r} 1273 \cdot 8241 \cdot 81 \dots \quad \underline{15} \\ 1274 \cdot 4612 \cdot 20 \dots \quad \underline{15} \\ 1275 \cdot 0985 \cdot 78 \dots \quad \underline{15} \end{array}$$

$$\begin{array}{r} 8242 \cdot 4080 \cdot 0 \dots \\ 8246 \cdot 5300 \cdot 2 \dots \quad \underline{15} \end{array}$$

$$\begin{array}{r} 25500 \text{ twice} \\ 8246 \text{ once} \\ \hline 33746 \end{array}$$

$$\begin{array}{r} 12 \cdot 750 \cdot 98578 \\ 8 \cdot 246 \cdot 53002 \\ \hline 20 \cdot 997 \cdot 51580 \text{ take} \\ 21 \cdot 000 \cdot 00000 \text{ from} \\ \hline 33746) 2 \cdot 48420 (\downarrow 0,0,0,0,7,3,6,1 \\ 2 \cdot 36222 \\ \hline 121980 \\ 101238 \\ \hline 207420 \\ 202476 \\ \hline 49440 \\ 33746 \end{array}$$

$$\therefore x = 2 \downarrow 0,3,0,5,7,3,6,1,$$

The last four figures being found by common division.

$$\therefore x = 2 \cdot 06178427, \text{ true to the last figure.}$$

2. Extract a root of the equation $x^3 + 2x^2 - 23x = 70$.

Substitute 5 for x , and the result approaches 70.

+	+	minus
125'00000000	50'00000000	115'00000000

3 times	37	5
2 times	10	0
	47	5
minus once	11	5
	36	

12	5'00000000
5	0'00000000
17	5'00000000
11	5'00000000
6	0'00000000 take
7	0'00000000 from
36)	10'00000000 (↓ 0,2,

12	50	00	00	00
13	13	76	25	63
13	26	90	01	89
	51			
	11			

50	00	00	00	0
52	03	02	00	5

11	5'0	00	00	00
11	7	3	11	50
	21			

3 times +	397	8
twice +	104	0
	+ 501	8
once -	117	3
	384	5

132	6900189
52	0302005
184	7202194
117	3115000
67	4087194 take
70	0000000 from
384)	2'5912806 (↓ 0,0,6

132	690	018	9	..
133	354	797	1	.. 151
134	022	906	0	.. 151
134	694	362	1	.. 151
135	098	849	4	.. 131

520	302	005
522	908	723
525	528	501
526	580	084

117	311	500	0	..
117	899	231	8	.. 151
118	017	131	0	.. 111

+ 4052	3 times —	135·0	9884 94
+ 1053	twice —	52·6	5800 84
5105		187·7	5685 78
— 1180	once —	118·0	1713 10
3925		69·7	3972 68 take
		70·0	0000 00 from
	3925)	2	6027 32 (↓ 0,0,0,6,

1350 9884 94 . .	5265 8008 4 . . .	1180 1713 10 . .
1351 6641 24 . . 15	5268 4342 6 . . . 15	1180 7615 14 . . 15
1352 3400 91 . . 15	5271 0690 1 . . . 15	1180 8796 90 . . 11
1353 0163 96 . . 15	5272 1232 7 . . . 12	
1353 4223 42 . . 13		

+ 40603 3 times	135·34	22342
+ 10544 twice	52·72	12327
51147	188·06	34669
— 11809 once	118·08	79690
39338	69·97	54979 take
	70·00	00000 from
	39338)	2
		45021 (↓ 0,0,0,0,6,2,3,

∴ $x = 5 \downarrow 0,2,6,6,6,2,3, = 5 \cdot 1345787$; the last three factors being found by common division.

3. *Extract a root of the biquadratic equation*

$$x^4 - 3x^2 + 75x = 1000.$$

If 8 be substituted for x , the result will approach 10000 near enough to commence operating.

4	2	1
plus	minus	plus
4096	192	600
	A A	

$$\begin{array}{r} 16384 \\ 384 \\ \hline \end{array}$$

4 times
twice

$$\begin{array}{r} 4|096 \\ |192 - \\ \hline \end{array}$$

$$\begin{array}{r} 16000 \\ 600 \\ \hline \end{array}$$

once

$$\begin{array}{r} 3|904 \\ |600 + \\ \hline \end{array}$$

$$\begin{array}{r} 16600 \\ \hline \end{array}$$

$$\begin{array}{r} 4|504 \text{ take} \\ 10|000 \text{ from} \\ \hline \end{array}$$

$$16600 \) \ 5|496 \quad (\downarrow 2,$$

plus

minus

$$\begin{array}{r} 4|0|9|6|0|0|0|0|0|0|0 \\ 6|5|9|6|6|4|8|9|6|0|0 \\ 8|7|8|0|1|3|9|7|6|5|8 \\ \hline \end{array} \quad \begin{array}{l} \downarrow 5, \\ \downarrow 3, \end{array}$$

$$\begin{array}{r} 1|9|2|0|0|0|0|0|0|0 \\ 2|8|1|1|0|7|2|0|0|0 \\ \hline \end{array} \quad \downarrow 4,$$

plus

$$\begin{array}{r} 6|0|0|0|0|0|0|0|0|0 \\ 7|2|6|0|0|0|0|0|0|0 - \downarrow 2. \end{array}$$

$$\begin{array}{r} 351|2 + \\ 5|6 - \\ \hline \end{array}$$

4 times
twice

$$\begin{array}{r} 87|80|1397658 \\ 2|81|1072000 - \\ \hline \end{array}$$

$$\begin{array}{r} 345|6 \\ 7|3 + \\ \hline \end{array}$$

once

$$\begin{array}{r} 84|99|0325658 \\ 7|26|0000000 + \\ \hline \end{array}$$

$$\begin{array}{r} 352|9 \\ \hline \end{array}$$

$$\begin{array}{r} 92|25|0325658 \text{ take} \\ 100|00|0000000 \text{ from} \\ \hline \end{array}$$

$$353 \) \ 7|74|9674342 \quad (0,2,$$

plus

minus

plus

$$\begin{array}{r} 87|80|13|97|65|8 . \\ 92|28|01|51|35|2 . \underline{51} \\ 95|07|63|32|21|8 . \underline{31} \end{array}$$

$$\begin{array}{r} 28|11|07|20|00 \\ 29|25|21|27|95 \underline{41} \end{array}$$

$$\begin{array}{r} 72|60|00|00|00 \\ 74|05|92|60|00 \underline{21} \end{array}$$

$$\begin{array}{r} 3803| + \\ 59| - \\ \hline \end{array}$$

4 times
twice

$$\begin{array}{r} 950|7|6332218 + \\ 29|2|5212795 - \\ \hline \end{array}$$

$$\begin{array}{r} 3744| \\ 74| + \\ \hline \end{array}$$

once

$$\begin{array}{r} 921|5|1119423 \\ 74|0|5926000 + \\ \hline \end{array}$$

$$\begin{array}{r} 3818 \\ \hline \end{array}$$

$$\begin{array}{r} 995|5|7045423 \text{ take} \\ 1000|0|0000000 \text{ from} \\ \hline \end{array}$$

$$3818 \) \ 4|4|2954577 \quad (\downarrow 0,0,1,$$

plus	minus	plus
950 763 322 18 .	292 521 279 5 . .	740 592 600 0 . .
954 572 083 85 . <u>14</u>	293 106 614 6 . . <u>12</u>	741 333 192 6 . . <u>11</u>

3818 3 +	4 times	954 57208385 +
58 6 -	twice	29 31066146 -
3759 7		925 26142239
74 1 +	once	74 13331926 +
3833 8		999 39474165 take
		1000 00000000 from

$$3834) \quad |6.0525835 \quad (\downarrow 0,0,0,1,$$

plus	minus	plus
9545 7208 385 .	2931 0661 46 . .	7413 3319 26 . .
9549 5396 995 . <u>14</u>	2931 6523 88 . . <u>12</u>	7414 0732 59 . . <u>11</u>

38198 +	4 times	9549 5396995 +
586 -	twice	293 1652388 -
37612		9256 3744607
741 +	once	741 4073259 +
38353		9997 7817866 take
		10000 00000000 from

$$38353) \quad 2.2182134 \quad (\downarrow 0,0,0,0,5,7,8,3,$$

$$\therefore x = 8 \downarrow 2,2,1,1,5,7,8,3 = 9.8860027 ;$$

the last four factors, 5, 7, 8, 3, were obtained by common division.

It may be observed that the work did not commence by operating with the first figure of the root, nor was any notice taken of the absent term of the equation.

4. *Extract a root of the equation*

$$x^5 + 4x^4 - 2x^3 + 10x^2 - 2x = 962.$$

If 3 be substituted for x , the number 962 will be approached, for

$$243 + 324 - 54 + 90 - 6 = 597.$$

$$\begin{array}{rcl}
 12|15 + & 5 \text{ times} & + 2|43 \\
 12|96 + & 4 \text{ times} & + 3|24 \\
 \hline
 25|11 & & 5|67 \\
 1|62 - & 3 \text{ times} & - 5|54 \\
 \hline
 23|49 & & 5|13 \\
 1|80 + & 2 \text{ times} & + 9|90 \\
 \hline
 25|29 & & 6|03 \\
 |6 - & 1 \text{ time} & - 6|6 \\
 \hline
 25|23 & & 5|97 \text{ take} \\
 & & 9|62 \text{ from}
 \end{array}$$

divisor 2523) 3|65 $\downarrow 1,$

$$\begin{array}{rcl}
 \text{plus} & & \text{plus} \\
 2|4|3|0|0|0|0|0|0 & & 3|2|4|0|0|0|0|0|0 \\
 3|9|1|3|5|3|9|3|0|0 & \downarrow 5, & 4|7|4|3|6|8|4|0|0|0 & \downarrow 4, \\
 \text{minus} & & \text{plus} \\
 5|4|0|0|0|0|0|0|0 & & 9|0|0|0|0|0|0|0|0 \\
 7|1|8|7|4|0|0|0|0 & \downarrow 3, & 10|8|9|0|0|0|0|0|0 & \downarrow 2, \\
 & & \text{minus} \\
 & & 6|0|0|0|0|0|0|0|0 \\
 & & 6|6|0|0|0|0|0|0|0 & \downarrow 1,
 \end{array}$$

$$\begin{array}{rcl}
 195|68 + & 5 \text{ times} & + 39|1'3|539300 \\
 189|74 + & 4 \text{ times} & + 47|4'3|684000 \\
 \hline
 385|42 & & 86|5'7|223300 \\
 21|56 - & 3 \text{ times} & - 7|1'8|740000 \\
 \hline
 363|86 & & 79|3'8|483300 \\
 21|78 + & 2 \text{ times} & + 10|8'9|000000 \\
 \hline
 385|64 & & 90|2'7|483300 \\
 66 - & 1 \text{ time} & - 6|6|000000 \\
 \hline
 384|08 & & 89|6'1|483300 \text{ take} \\
 & & 96|2'0|000000 \text{ from}
 \end{array}$$

divisor 384|98) 6|5'8|516700 $\downarrow 0,1,$

In these early examples, the array of figures employed to find the divisor, and the next factor to be operated with, may be omitted; for when the method is understood, the determination of each successive factor requires but little calculation.

$$\begin{array}{r} \text{plus} \\ 39 \overline{) 13} \overline{) 53} \overline{) 93} \overline{) 00} \\ 41 \overline{) 13} \overline{) 16} \overline{) 91} \overline{) 35} \quad \underline{5} \end{array}$$

$$\begin{array}{r} \text{minus} \\ 71 \overline{) 87} \overline{) 40} \overline{) 00} \overline{) 0} \cdot \\ 74 \overline{) 05} \overline{) 18} \overline{) 54} \overline{) 1} \cdot \quad \underline{3} \end{array}$$

$$\begin{array}{r} \text{plus} \\ 47 \overline{) 43} \overline{) 68} \overline{) 40} \overline{) 00} \\ 49 \overline{) 36} \overline{) 29} \overline{) 65} \overline{) 92} \quad \underline{4} \end{array}$$

$$\begin{array}{r} \text{plus} \\ 10 \overline{) 89} \overline{) 00} \overline{) 00} \overline{) 00} \\ 11 \overline{) 10} \overline{) 88} \overline{) 89} \overline{) 00} \quad \underline{2} \end{array}$$

$$\begin{array}{r} \text{minus} \\ 66 \overline{) 00} \overline{) 00} \overline{) 00} \\ 66 \overline{) 66} \overline{) 00} \overline{) 00} \quad \underline{1} \end{array}$$

205 $\overline{) 658}$ +	5 times	+ 41 $\overline{) 131}$ $\overline{) 69135}$
197 $\overline{) 451}$ +	4 times	+ 49 $\overline{) 362}$ $\overline{) 96592}$
403 $\overline{) 109}$		90 $\overline{) 494}$ $\overline{) 65727}$
22 $\overline{) 215}$ -	3 times	- 7 $\overline{) 405}$ $\overline{) 18541}$
380 $\overline{) 894}$		83 $\overline{) 089}$ $\overline{) 47186}$
22 $\overline{) 217}$ +	2 times	+ 11 $\overline{) 108}$ $\overline{) 88900}$
403 $\overline{) 111}$		94 $\overline{) 198}$ $\overline{) 36086}$
666 -	1 time	- 6 $\overline{) 66}$ $\overline{) 60000}$
402 $\overline{) 445}$		93 $\overline{) 531}$ $\overline{) 76086}$ take
		96 $\overline{) 200}$ $\overline{) 00000}$ from

$$402 \) \quad 2 \overline{) 668} \overline{) 23914} \quad (\downarrow 0,0,6,$$

$$\begin{array}{r} \text{plus} \\ 411 \overline{) 316} \overline{) 913} \overline{) 5} \cdot \cdot \\ 413 \overline{) 377} \overline{) 615} \overline{) 4} \cdot \cdot \underline{151} \\ 415 \overline{) 448} \overline{) 641} \overline{) 4} \cdot \cdot \underline{151} \\ 417 \overline{) 530} \overline{) 043} \overline{) 3} \cdot \cdot \underline{151} \\ 419 \overline{) 621} \overline{) 873} \overline{) 0} \cdot \cdot \underline{151} \\ 421 \overline{) 724} \overline{) 182} \overline{) 8} \cdot \cdot \underline{151} \\ 423 \overline{) 837} \overline{) 025} \overline{) 1} \cdot \cdot \underline{151} \end{array}$$

$$\begin{array}{r} \text{plus} \\ 493 \overline{) 629} \overline{) 659} \overline{) 2} \cdot \cdot \\ 496 \overline{) 102} \overline{) 748} \overline{) 7} \cdot \cdot \underline{151} \\ 498 \overline{) 588} \overline{) 228} \overline{) 4} \cdot \cdot \underline{151} \\ 501 \overline{) 086} \overline{) 160} \overline{) 4} \cdot \cdot \underline{151} \\ 503 \overline{) 596} \overline{) 607} \overline{) 1} \cdot \cdot \underline{151} \\ 505 \overline{) 614} \overline{) 017} \overline{) 1} \cdot \cdot \underline{141} \end{array}$$

minus	plus
740 518 541	111 088 890 0 . .
744 228 546 151	111 645 446 5 . . 151
747 957 138 151	112 204 791 3 . . 151
751 704 411 151	112 429 313 1 . . 121
753 961 780 131	

minus
666 600 00 .
669 939 68 . 151
670 609 62 . 111

2119 2	5 times	+ 423 8370 251
2022 5	4 times	+ 505 6140 171
4141 7		929 4510 322
226 2	3 times	- 75 3961 780
3915 5		854 0548 542
224 9	2 times	+ 112 4293 131
4140 4		966 4841 673
6 7	1 time	- 6 7060 962
4133 7		959 7780 711 take
		962 0000 000 from

4134) 2|2219|289 (↓ 0,0,0,5,3,7, by
common division.

$$\therefore x = 3 \downarrow 1,1,6,5,3,7, = 3 \ 354849.$$

But for the purpose of rendering the process clear, this root might be found with less than one-third the figures exhibited.

5. Required a root of the equation

$$3 \cdot 01416x^5 - 28 \cdot 233x^4 + 923 \cdot 7x^3 + 1234x^2 - 1862x = 1609149128.$$

It is easily found that the root is between 10 and 100; 40 will reduce the final number considerably, but 50 approaches much nearer. However, the operation may be commenced with either 50 or 40. With 50 the following result is found:—

$\begin{array}{r} + 46 0 \\ - 7 0 \\ \hline 39 0 \\ + 4 6 \\ \hline 43 6 \\ + 0 \\ \hline 43 6 \\ - 0 \\ \hline 43 6 \end{array}$	5 times 4 times 3 times 2 times 1 time	$\begin{array}{r} + 9 41925000 . \\ - 1 76456250 . \\ \hline 7 65468750 . \\ + 1 15462500 . \\ \hline 8 80931250 . \\ + 3085000 . \\ \hline 8 84016250 \\ - 93100 \\ \hline 8 83923150 \text{ take} \\ 16 09149128 \text{ from} \end{array}$
---	--	--

$$44) 7|25225978 \quad (\downarrow 1,$$

plus	minus
$\begin{array}{r} 9 41 92 50 00 \\ 15 16 97 96 32 \downarrow 5, \end{array}$	$\begin{array}{r} 1 7 6 4 5 6 2 5 0 \\ 2 5 8 3 4 9 5 9 6 \downarrow 4, \end{array}$

plus	plus
$\begin{array}{r} 11 5 4 6 2 5 0 0 \\ 15 3 6 8 0 5 8 8 \downarrow 3, \end{array}$	$\begin{array}{r} 3 0 8 5 0 0 0 \\ 3 7 3 2 8 5 0 \downarrow 2, \end{array}$

minus

$$\begin{array}{r} 9|3|1|0|0 \\ 10|2|4|1|0 \downarrow 1, \end{array}$$

$\begin{array}{r} + 75 85 \\ - 10 32 \\ \hline 65 53 \\ + 4 62 \\ \hline 70 15 \\ + 8 \\ \hline 70 23 \\ - 0 \\ \hline \end{array}$	5 times 4 times 3 times 2 times 1 time	$\begin{array}{r} + 15 17 \text{ Omitting six figures.} \\ - 2 58 \text{ , ,} \\ \hline 12 59 \\ + 1 54 \\ \hline 14 13 \\ + 4 \\ \hline 14 17 \\ - 0 \\ \hline \end{array}$
---	--	--

$$\begin{array}{r} 14|17 \text{ take} \\ 16|09 \text{ from} \end{array}$$

$$70) 1|92 (\downarrow 0,2,$$

plus	minus
$\begin{array}{r} 15 16 97 96 32 \\ 15 94 36 08 40 \underline{51} \\ 16 75 68 92 67 \underline{51} \end{array}$	$\begin{array}{r} 25 83 49 59 6 . \\ 27 15 28 02 2 . \underline{51} \\ 27 97 55 59 3 . \underline{31} \end{array}$

$$\begin{array}{r}
 \text{plus} \\
 15|36|80|58|8. \\
 16|15|19|84|3. \quad \underline{51} \\
 16|31|35|04|1. \quad \underline{11}
 \end{array}$$

$$\begin{array}{r}
 \text{plus} \\
 37|32|85|0. \\
 38|84|41|9. \quad \underline{41}
 \end{array}$$

$$\begin{array}{r}
 \text{minus} \\
 10|24|10| \\
 10|44|68| \quad \underline{21}
 \end{array}$$

$$\begin{array}{r}
 + 83785 \\
 - 11192 \\
 \hline
 \end{array}$$

5 times
4 times

$$\begin{array}{r}
 + 72593 \\
 + 4893 \\
 \hline
 \end{array}$$

3 times

$$\begin{array}{r}
 + 77486 \\
 + 76 \\
 \hline
 \end{array}$$

2 times

$$\begin{array}{r}
 - 77562 \\
 - 1 \\
 \hline
 \end{array}$$

1 time

$$\begin{array}{r}
 77561
 \end{array}$$

$$\begin{array}{r}
 + 167|57 \text{ Omitting five figures.} \\
 - 27|98 \quad \text{,,} \quad \text{,,}
 \end{array}$$

$$\begin{array}{r}
 + 139|59 \\
 + 16|31 \quad \text{,,} \quad \text{,,}
 \end{array}$$

$$\begin{array}{r}
 + 155|90 \\
 + 38 \quad \text{,,} \quad \text{,,}
 \end{array}$$

$$\begin{array}{r}
 - 156|28 \\
 - 1 \quad \text{,,} \quad \text{,,}
 \end{array}$$

$$\begin{array}{r}
 156|27 \text{ take} \\
 160|91 \text{ from}
 \end{array}$$

$$776) \quad 4|64 \quad (\downarrow 0,0,5,$$

$$\begin{array}{r}
 \text{plus} \\
 167|568|926|7. . \\
 168|408|448|7. . \quad \underline{151} \\
 169|252|176|7. . \quad \underline{151} \\
 170|100|131|8. . \quad \underline{151} \\
 170|952|335|2. . \quad \underline{151} \\
 171|808|808|1. . \quad \underline{151}
 \end{array}$$

$$\begin{array}{r}
 \text{minus} \\
 279|755|593 \\
 281|157|172 \quad \underline{151} \\
 282|565|773 \quad \underline{151} \\
 283|981|431 \quad \underline{151} \\
 285|404|181 \quad \underline{151}
 \end{array}$$

$$\begin{array}{r}
 \text{plus} \\
 163|135|041 \\
 163|952|349 \quad \underline{151} \\
 164|773|753 \quad \underline{151} \\
 165|599|272 \quad \underline{151}
 \end{array}$$

$$\begin{array}{r}
 \text{plus} \\
 388|441|9. . \\
 390|388|0. . \quad \underline{151} \\
 392|343|8. . \quad \underline{151}
 \end{array}$$

$$\begin{array}{r}
 \text{minus} \\
 104|468 \\
 104|991 \quad \underline{151}
 \end{array}$$

+ 8590 44	5 times	+ 1718 0880 81
- 1141 62	4 times	- 285 4041 81
7448 82		1432 6839 00
+ 496 80	3 times	+ 165 5992 72
7945 62		1598 2831 72
+ 7 85	2 times	+ 3 9234 38
7953 47		1602 2066 10
- 10	1 time	- 1049 91
7953 37		1602 1016 19 take
		1609 1491 28 from

$$7953) \quad 7|0475|09 \quad (\downarrow 0,0,0,8,8,6,$$

$$\therefore x = 50 \downarrow 1,2,5,8,8,6, = 56.43657$$

6. Find the value of x in the equation

$$x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x = 654321.$$

If 8 be substituted for x , the result will be as follows:

15 7	6 times	2 62144 ~ x^6
3 2	5 times	65536 ~ $2x^5$
5	4 times	12288 ~ $3x^4$
.	3 times	2048 ~ $4x^3$
.	2 times	320 ~ $5x^2$
.	1 time	48 ~ $6x$
19 4		3 42 ... from
		6 54 ... take
		19) 3 12 \quad (\downarrow 1,

plus

$$\begin{array}{r} 2|6|2|1|4|4|0|0|0 \\ 4|2|2|1|8|5|5|3|3 \downarrow 5, \\ 4|6|4|4|0|4|0|8|6 \downarrow 1, \end{array}$$

plus

$$\begin{array}{r} 6|5|5|3|6|0|0|0 \\ 10|5|5|4|6|3|8|3 \downarrow 5, \end{array}$$

plus

$$\begin{array}{r} 1|2|2|8|8|0|0|0 \\ 1|7|9|9|0|8|6|1 \downarrow 4, \end{array}$$

plus

$$\begin{array}{r} 2|0|4|8|0|0|0 \\ 2|7|2|5|8|8|8 \downarrow 3, \end{array}$$

plus

$$\begin{array}{r} 3|2|0|0|0|0 \\ 3|8|7|2|0|0 \downarrow 2, \end{array}$$

plus

$$\begin{array}{r} 4|8|0|0|0 \\ 5|2|8|0|0 \downarrow 1, \end{array}$$

B B

omitting decimals

$$\begin{array}{r} 278 \\ 52 \\ 7 \\ 1 \\ . \\ . \\ \hline 34 \cdot \end{array}$$

6 times
5 times
4 times
3 times
2 times
1 time

$$\begin{array}{r} 46 \\ 10 \\ 1 \\ 2725 \\ 387 \\ 52 \\ \hline \end{array}$$

$$\begin{array}{r} 59 \\ 65 \\ \hline \end{array} \begin{array}{l} 1 \dots \text{take} \\ 4 \dots \text{from} \end{array}$$

$$34) 63 \dots (\downarrow 0,1,$$

$$\begin{array}{r} 46 \\ 48 \\ 49 \end{array} \begin{array}{r} 44 \\ 80 \\ 29 \end{array} \begin{array}{r} 04 \\ 93 \\ 74 \end{array} \begin{array}{r} 08 \\ 36 \\ 29 \end{array} \begin{array}{r} 6 \\ 1 \\ 5 \end{array} \begin{array}{r} . \\ . \\ . \end{array} \begin{array}{r} . \\ 51 \\ 11 \end{array}$$

$$\begin{array}{r} 10 \\ 11 \end{array} \begin{array}{r} 55 \\ 09 \end{array} \begin{array}{r} 46 \\ 30 \end{array} \begin{array}{r} 38 \\ 30 \end{array} \begin{array}{r} 3 \\ 8 \end{array} \begin{array}{r} . \\ . \end{array} \begin{array}{r} . \\ 51 \end{array}$$

$$\begin{array}{r} 17 \\ 18 \end{array} \begin{array}{r} 99 \\ 72 \end{array} \begin{array}{r} 08 \\ 13 \end{array} \begin{array}{r} 61 \\ 61 \end{array} \begin{array}{r} . \\ 41 \end{array}$$

$$\begin{array}{r} 27 \\ 28 \end{array} \begin{array}{r} 25 \\ 08 \end{array} \begin{array}{r} 88 \\ 48 \end{array} \begin{array}{r} 8 \\ 6 \end{array} \begin{array}{r} . \\ . \end{array} \begin{array}{r} . \\ 31 \end{array}$$

$$\begin{array}{r} 38 \\ 39 \end{array} \begin{array}{r} 72 \\ 49 \end{array} \begin{array}{r} 00 \\ 83 \end{array} \begin{array}{r} . \\ 21 \end{array}$$

$$\begin{array}{r} 52 \\ 53 \end{array} \begin{array}{r} 80 \\ 32 \end{array} \begin{array}{r} 0 \\ 8 \end{array} \begin{array}{r} . \\ . \end{array} \begin{array}{r} . \\ 11 \end{array}$$

$$\begin{array}{r} 294 \\ 55 \\ 7 \\ 1 \\ . \\ . \\ \hline 357 \end{array}$$

$$\begin{array}{r} 49 \\ 11 \\ 1 \\ 2808 \\ 394 \\ 53 \\ \hline \end{array}$$

$$\begin{array}{r} 62 \\ 65 \\ \hline \end{array} \begin{array}{r} 5880 \\ 4321 \end{array}$$

$$36) 28441 (\downarrow 0,0,7,$$

$$\begin{array}{r} 492 \\ 514 \end{array} \begin{array}{r} 974 \\ 109 \end{array} \begin{array}{r} 295 \\ 379 \end{array} \begin{array}{r} . \\ 421 \end{array}$$

$$\begin{array}{r} 110 \\ 114 \end{array} \begin{array}{r} 930 \\ 879 \end{array} \begin{array}{r} 308 \\ 605 \end{array} \begin{array}{r} . \\ 351 \end{array}$$

$$\begin{array}{r} 187 \\ 192 \end{array} \begin{array}{r} 213 \\ 526 \end{array} \begin{array}{r} 61 \\ 97 \end{array} \begin{array}{r} . \\ 281 \end{array}$$

$$\begin{array}{r} 280 \\ 286 \end{array} \begin{array}{r} 848 \\ 805 \end{array} \begin{array}{r} 6 \\ 8 \end{array} \begin{array}{r} . \\ . \end{array} \begin{array}{r} . \\ 211 \end{array}$$

$$\begin{array}{r} 394 \\ 400 \end{array} \begin{array}{r} 983 \\ 549 \end{array} \begin{array}{r} . \\ 141 \end{array}$$

$$\begin{array}{r} 533 \\ 537 \end{array} \begin{array}{r} 28 \\ 02 \end{array} \begin{array}{r} . \\ 71 \end{array}$$

308	6 times	514 109 379
57	5 times	114 879 605
8	4 times	19 252 697
1	3 times	2 868 058
.	2 times	400 549
.	1 time	53 702
374		651 563'990 take 654 321'000 from

37) 2|757'010 (↓ 0,0,0,7,

5141|0937|9 ... 1148|7960|5 ... 1925|2697
5162|7307|0 ... 42| 1152|8236|9 ... 35| 1930|6678 28|

2868|058 . 4005|49 . 5370|2 ...
2874|087 . 21| 4011| .. 14| 5374|0 ... 7|

3097638420 5162|73 070
576411845 1152|82 369
77226712 193|06 678
8622261 28|74 087
802220 4|01 110
53740 53 740

3760755198 6541|91'054 take
6543|21'000 from

3761) 1|29 946 (↓ 0,0,0,0,3,4,5,5,

∴ $x = 8 \downarrow 1,1,7,7,3,4,5,5, = 8.95697958.$

To reduce a number as 492974295 to 514109379 42| has been explained in the first part of this work; but the result may be found in one operation, which may be arranged as follows:

492|974|295 = 1
20|704|919 = 42 = a
424|451 = 41 × a ÷ 2 = b
5|659 = 40 × b ÷ 3 = c
55 = 39 × c ÷ 4 = d

514 109 379 42|

Calculation, to find the line a :

$$\begin{array}{r}
 492974 \cdot 259 \\
 \underline{42} \\
 985948 \ 518 \\
 19718970 \ 36 \\
 \hline
 20704918 \cdot 878 \\
 20704919 \text{ set down}
 \end{array}
 \begin{array}{c}
 \begin{array}{ccc}
 & 0 & \\
 6 & \times & 6 \\
 & 0 &
 \end{array}
 \end{array}$$

To find the line b :

$$\begin{array}{r}
 2 \) \ 20704 \cdot 919 \\
 \hline
 10352 \cdot 46 \\
 \underline{41} \\
 10352 \ 46 \\
 414098 \ 4 \\
 \hline
 424450 \cdot 86 \\
 424451 \text{ set down.}
 \end{array}
 \begin{array}{c}
 \begin{array}{ccc}
 & 6 & \\
 3 & \times & 5 \\
 & 6 &
 \end{array}
 \end{array}$$

This direct method can be proved by casting out nines, which has an advantage, as the result may be relied on as correct. In working these examples, the most obvious arithmetical abridgements are not attended to, in order that this new method of extracting the roots of all equations may be presented without any disguise. The accomplished calculator can, at his pleasure, introduce many expedients to reduce numerical labour.

7. Find the value of x in the equation

$$421x^7 + 356 \cdot 2x^5 - 548x^4 - 298x = 9876543210.$$

Although it is not absolutely necessary to commence operating with the first figure of the root, yet the nearer we approach this number to commence with, the easier the required root is extracted. This example, set down at random, three of the terms absent, the coefficients large and irregular without any particular arrangement, will tend to establish the great power of this new and general method of solving numerical equations.

It will be found that $x = 10$, tends to approach the absolute number 9876543210, sufficiently near, to commence operating.

29	+ 7 times	4 210000000	+ $421x^7$
.	+ 5 times	35620000	+ $356 \cdot 2x^5$
.	- 4 times	4 245620000	- $548x^4$
.	- 1 time	4 240140000	- $2983x$
29		4 240110170	take
		9 876543210	from

$$29 \) \ 56 \dots\dots\dots (\downarrow 1,$$

$$\begin{array}{r} 4 \ 2 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \\ 6 \ 7 \ 8 \ 0 \ 2 \ 4 \ 7 \ 1 \ 0 \ 0 \ \downarrow 5, \\ 8 \ 2 \ 0 \ 4 \ 0 \ 9 \ 8 \ 9 \ 9 \ 1 \ \downarrow 2, \end{array}$$

$$\begin{array}{r} 3 \ 5 \ 6 \ 2 \ 0 \ 0 \ 0 \ 0 \\ 5 \ 7 \ 3 \ 5 \ 1 \ 7 \ 6 \ 2 \ \downarrow 5, \end{array}$$

$$\begin{array}{r} 5 \ 4 \ 8 \ 0 \ 0 \ 0 \ 0 \\ 8 \ 0 \ 2 \ 3 \ 2 \ 6 \ 8 \ \downarrow 4, \end{array}$$

$$\begin{array}{r} 2 \ 9 \ 8 \ 3 \ 0 \\ 3 \ 2 \ 8 \ 1 \ 3 \ \downarrow 1, \end{array}$$

57	+ 7 times	8 204098991
.	+ 5 times	57351762
.	- 4 times	8 261450753
.	- 1 time	8 253427485

57		8 253394672	take
		9 876543210	from

$$57 \) \ 162 \dots\dots\dots (\downarrow 0,2,$$

$$\begin{array}{r} 82 \ 04 \ 09 \ 89 \ 91 \\ 94 \ 30 \ 40 \ 02 \ 33 \ \underline{14} \end{array}$$

$$\begin{array}{r} 57 \ 35 \ 17 \ 62 \\ 63 \ 35 \ 20 \ 24 \ \underline{10} \end{array}$$

$$\begin{array}{r} 80 \ 23 \ 26 \ 8 \\ 86 \ 88 \ 04 \ 8 \ \underline{8} \end{array}$$

$$\begin{array}{r} 32 \ 81 \ 3 \\ 33 \ 47 \ 2 \ \underline{2} \end{array}$$

6601	+ 7 times	943 0	
32	+ 5 times	6 3	
6633		949 3	
3	- 4 times	8	
6630		948 5	
.	- 1 time		
6630		948 5 take	
		987 6 from	
		66) 39 1	(↓ 0,0,5,

943 040 023 4 . .	633 520 25 .
976 613 754 9 . . 35	649 552 71 . 25
868 804 8 . .	334 72 .
886 346 9 . . 20	336 39 . 5

68363	+ 7 times	976 6137549	
325	+ 5 times	6 4955271	
68688		983 1092820	
35	- 4 times	8863469	
68653		982 2229351	
.	- 1 time	33639	
68653		982 2195712 take	
		987 6543210 from	
		68653) 5 4347498	(↓ 0,0,0,7,9,2,

$$\therefore x = 10 \downarrow 1,2,5,7,9,2, = 11'286252.$$

8. Find the value of x in the equation

$$x^2 - 120x - 100x^{\frac{1}{2}} + 999x^{\frac{1}{2}} = 91000.$$

The value of x lies somewhere between 3 and 4 hundred; commencing with $x = 300$, then

180	twice	+ 90 000	= x^2
36	once	- 36 000	= - 120x
144			
1	half	- 54 000	
143			
2	third	+ 52 267.9492	
145			
		+ 6 687.6352	= + 999x ¹
		58 955.5844 take	
		91 000.0000 from	

$$145 \) \ 32|044\ 4156 \ (\downarrow 0,22,$$

$$29|0$$

$$3\ 04$$

$$2\ 90$$

The nearest less number to 22 that can be divided by 2 and 3, or by 6, is 18.

$$90|00\ 0'0|00|0 .$$

$$128|76\ 9\ 1|90|5 . \downarrow 0,36,$$

$$36|00\ 0'0|00|0 .$$

$$43|06\ 1'3|09|1 . \downarrow 0,18,$$

$$17|32\ 0'5|08|0 .$$

$$18|94\ 3'1|84|5 . \downarrow 0,9,$$

$$66|87\ 6'3|52|0 .$$

$$70|99\ 0'5|95|2 . \downarrow 0,6,$$

$$25754$$

$$4306$$

$$\hline 21448$$

$$99$$

$$\hline 21349$$

$$236$$

$$\hline 21585$$

twice
once

$$+ 12876|9\ 1905$$

$$- 4306|1\ 3091$$

$$\hline 8570|7\ 8812$$

$$- 189|4\ 3185$$

$$\hline 8381|3\ 5629$$

$$709|9\ 0595$$

$$\hline 9091|2\ 6224 \text{ take}$$

$$9100|0\ 0000 \text{ from}$$

$$2188 \) \ 8|7\ 3776 \ (\downarrow 0,0,0,0,40,$$

The nearest less number to 40 divisible by 6 is 36; therefore,

$$12876|9\ 1905 = 1$$

$$9|2\ 7138 = 72 = a$$

$$658 = 71 \times a \div 2$$

$$12886\ 1'9701$$

$$43061|3091 . = 1$$

$$15|5021 . = 36 = a$$

$$27 . = 35 \times a \div 2$$

$$43076\ 8139$$

$$\begin{array}{r|l} 18943 & 185 \dots 1 \\ 3 & 410 \dots 18 \end{array}$$

$$\hline 1894 \cdot 6595$$

$$\begin{array}{r|l} 70990 & 595 \dots = 1 \\ 8 & 519 \dots = 12 = a \\ & 1 \dots = 13 \times a \div 2 \end{array}$$

$$\hline 7099 \cdot 9115$$

$$\begin{array}{r} 25772 \text{ twice } + 12886 | 1 \cdot 9701 \\ 4308 \text{ once } - 4307 | 6 \cdot 8139 \end{array}$$

$$\begin{array}{r} 21464 \\ 99 \text{ half } - 8578 | 5 \cdot 1562 \\ 189 | 4 \cdot 6595 \end{array}$$

$$\begin{array}{r} 21365 \\ 236 \text{ third } + 8389 | 0 \cdot 4967 \\ 709 | 9 \cdot 9115 \end{array}$$

$$\begin{array}{r} 21601 \\ 9099 | 0 \cdot 4082 \text{ take} \\ 9100 | 0 \cdot 0000 \text{ from} \end{array}$$

$$216) \quad 195918 \quad (\downarrow 0,0,0,0,0,444, = \downarrow \boxed{444},$$

$$\begin{aligned} \therefore x &= 300 \downarrow 0,18,0,0,36,444, = 358 \cdot 98937 \\ &= 300 \downarrow 0,18,0,0,36,42,24, \end{aligned}$$

This value of x is given in a form so that the factors to the right of $\downarrow$ are divisible by 6; the work is easily proved by continuing the process two more steps.

$$\begin{aligned} x &= 300 \downarrow 0,18, \quad 0, \quad 0,36,42, \quad 24, \\ &= \downarrow 0, \quad 0,180, \quad 0,36,30, \quad 102, \\ &= \downarrow 0, \quad 0, \quad 0,1800,36,30, \quad 822, \\ &= \downarrow 0, \quad 0, \quad 0, \quad 0, \quad 0,1795278, = \downarrow \boxed{1795278}, \\ &= \downarrow 1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 842083, \\ &= \downarrow 1, \quad 8, \quad 0, \quad 0, \quad 0, \quad 45979, \\ &= \downarrow 1, \quad 8, \quad 4, \quad 0, \quad 0, \quad 5995, \\ &= \downarrow 1, \quad 8, \quad 4, \quad 5, \quad 9, \quad 9, \quad 5, \end{aligned}$$

These reductions are instantly made by the following equalities so often employed:—

$$\downarrow 1, = \downarrow 0,10, \bar{4}, \bar{1}, \bar{9}, \bar{5}, \bar{1}, = \downarrow \bar{953195},$$

$$\downarrow 0,1, = \downarrow 0, 0,10, 0, \bar{4}, \bar{4}, \bar{7}, = \downarrow \bar{99513},$$

$$\downarrow 0,0,1, = \downarrow 0, 0, 0,10, 0,0,\bar{4}, = \downarrow \bar{9996},$$

$$\downarrow 0,0,0,1, = \downarrow 0, 0, 0, 0,10,0,0, = \downarrow \bar{1000},$$

The succeeding equations of condition are also easily adduced,

$$\downarrow 6, = \downarrow \bar{5719170},$$

$$\downarrow 0,6, = \downarrow \bar{597078},$$

$$\downarrow 0,0,6, = \downarrow \bar{59976},$$

$$\downarrow 0,0,0,6, = \downarrow \bar{6000},$$

$$\downarrow 0,0,0,0,6, = \downarrow \bar{600},$$

$$\downarrow 0,0,0,0,0,6, = \downarrow \bar{60},$$

$$\downarrow 0,0,0,0,0,0,6, = \downarrow \bar{6},$$

Hyperbolic logarithms of $\downarrow 1,; \downarrow 2,; \downarrow 3,; \&c.$ of $\downarrow 0,1,; \downarrow 0,2,; \downarrow 0,3,; \&c. \&c.$ are easily determined, for putting $c = 10000978$, then, because $\epsilon = \downarrow \bar{10000978},$

$$\log. \downarrow 1, \quad \downarrow \bar{953195}, \div c = \cdot 0953102$$

$$\log. \downarrow 0,1, \quad \downarrow \bar{99513}, \div c = \cdot 0099503$$

$$\log. \downarrow 0,0,1, \quad \downarrow \bar{9996}, \div c = \cdot 0009995$$

$$\log. \downarrow 0,0,0,1, \quad \downarrow \bar{1000}, \div c = \cdot 0001000$$

$$\log. \downarrow 0,0,0,0,1, \downarrow \bar{100}, \div c = \cdot 0000100$$

&c.

&c.

&c.

$$\therefore \log. \downarrow 12, = \cdot 0953102 \times 12 = 1\cdot 1437224$$

$$\log. \downarrow 0,9, = \cdot 0099503 \times 9 = \cdot 0895527$$

$$\log. \downarrow 0,0,5, = \cdot 0009995 \times 5 = \cdot 0049975$$

$$\log. \downarrow 0,0,0,1, = \cdot 0001000 \times 1 = \cdot 0001000$$

$$\log. \downarrow 0,0,0,0,0,1, = \cdot 0000010 \times 1 = \cdot 0000010$$

$$\log. \downarrow 0,0,0,0,0,0,5, = \cdot 0000001 \times 5 = \underline{\underline{5}}$$

$$\underline{\underline{1\cdot 2383742}}$$

$$\therefore \log. \downarrow 12,9,5,1,0,1,5, = 1.2383742$$

$$3.4499997 = \downarrow 12,9,5,1,0,1,5,$$

Such transformations and equalities as these will be found useful, especially in solving problems like the following:—

9. Find the value of x in the equation

$$x^m = e^{ax}$$

$$m \log. x = ax$$

$\therefore \frac{\log. x}{x} = \frac{m}{a} =$ suppose $.358949$ since a and m are known quantities.

Very few trials will show that the required number is between $\downarrow 12$, and $\downarrow 13$,

$$\begin{array}{r} 10000000 \\ 13000000 \\ 78000000 \\ 28600000 \\ 7150000 \\ 1287000 \\ 171600 \\ 17160 \\ 1297 \end{array}$$

$$3.4522712$$

$$\log. \downarrow 13, = 13 \times 0.953102 \\ = 1.2390326$$

$$\therefore \frac{1.2390326}{3.4522712} = .3589036$$

$$\begin{array}{r} 10000000 \\ 12000000 \\ 66000000 \\ 22000000 \\ 49500000 \\ 7920000 \\ 924000 \\ 79200 \\ 4950 \\ 227 \end{array}$$

$$3.13842844$$

$$\log \downarrow 12, = 12 \times .0953102 \\ = 1.1437224$$

$$\therefore \frac{1.1437224}{3.13842844} = .3633252$$

The number required will be found to be $\downarrow 12,9,5,1,0,1,6$, which is evidently nearer to $\downarrow 13$, than to $\downarrow 12$,

The nearest less digit found in solving the simple equation,

$$(3\cdot1|38|42|84|4 + 31384284 \times n) \times \cdot358949 \\ = 1\cdot1436224 + n \times \cdot0099503$$

for n gives $n = 9$. Many of these figures may be omitted in finding n ; in a similar way, 5, 1, 0, 6, may be ascertained, and ultimately

$$x = \downarrow 12,9,5,1,0,1,6, = 3\cdot45 \text{ nearly.}$$

Or the factors $\downarrow 0,9$, ; $\downarrow 0,0,5$, ; &c. may be found by division, for if n be made to represent the second factor, then the trial equation becomes

$$3\cdot1|38|42 + \cdot03138)n = (1\cdot1436224 + \cdot0099503)n \div \cdot358949 \\ = 3\cdot160 + \cdot02772 n \\ \therefore \cdot00366 n = \cdot0476$$

Consequently, n must be 9; for although $\cdot00366$ is contained in $\cdot0476$, more than 9 times, yet n may be taken for certain 9, because x lies between $\downarrow 12$, and $\downarrow 13$, but $\downarrow 12,10$, would exceed $\downarrow 13$,. However, by continuing the process one step further, the truth is established:

$$\begin{array}{rcl} 31|38|42|84|4 & \log. \downarrow 12, & = 1\cdot1437224 \\ 34|32|45|29|6 \cdot 91 & \log. \downarrow 0,9, & \cdot0895527 \\ & \therefore \log. \downarrow 12,9, & = 1\cdot2332751 \end{array}$$

$$\therefore \log. 3\cdot43245296 = 1\cdot2332751$$

$$\frac{1\cdot2332751}{3\cdot43245296} = \cdot3592985$$

Suppose m = the next factor, then

$$\begin{aligned} & 3\cdot43245296 + \cdot00343245 m \\ & = (1\cdot2332751 + \cdot0009995 m) \div \cdot358949 \\ & = 3\cdot435795 + \cdot00278452 m \end{aligned}$$

$$\therefore \cdot 00064793 m = \cdot 003342$$

$$\therefore m = 5.$$

To continue and advance the work another step :

$$\begin{array}{r|l} 343 & 245 & 296 \\ 344 & 964 & 957 & \underline{151} \end{array}$$

$$\log. \downarrow 12, = 1\cdot1437224$$

$$\log. \downarrow 0,9, = \cdot 0895527$$

$$\log. \downarrow 0,0,5, = \cdot 0049975$$

$$\therefore \log. \downarrow 12,9,5, = 1\cdot2382726$$

$$\frac{1\cdot2382726}{3\cdot44964957} = \cdot 3589560$$

Let r represent the next factor, then

$$3\cdot44964957 + \cdot 000344965 r$$

$$= (1\cdot2382726 + \cdot 0001000 r) \div \cdot 358949$$

$$= 3\cdot4497174 + \cdot 000278591 r$$

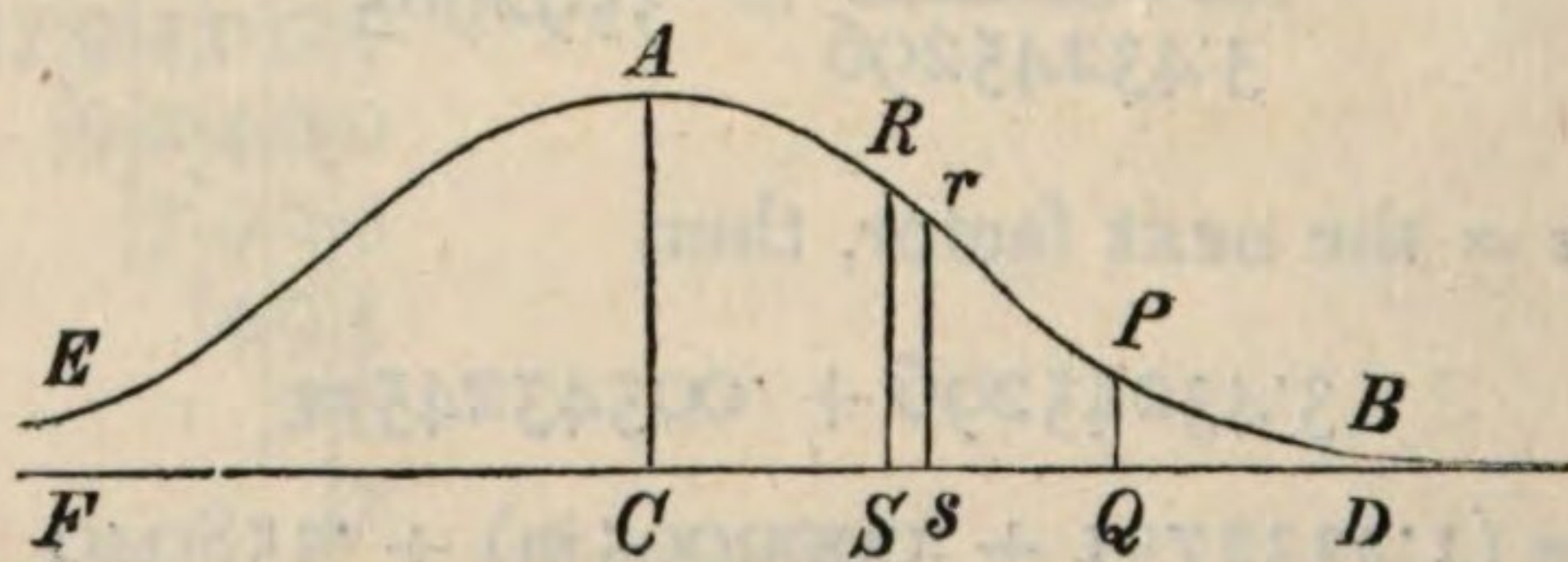
$$\therefore \cdot 000066374 r = \cdot 0000678$$

$$\therefore r = 101.$$

The three succeeding factors 1, 0, 1, may be determined in one operation ; the required number is approached rapidly when the first two factors become known. In this manner, the required number is found to be $\downarrow 12,9,5,1,0,1,6, = 3\cdot45$.

As in other cases, the details of the work of this example are given at great length, the more clearly to show that the factors comprising the required number are not guessed at, but determined by a well-defined law.

10. Let $y = \frac{2}{\sqrt{\pi}} e^{-x^2}$, be the equation of the curve $EARB$,



find the values of x and y , when the area $= \frac{1}{2}$.

By the integral calculus, it may be shown that

$$\int_0^x y \, dx = \frac{2}{\sqrt{\pi}} \int_0^x dx \left(1 - x^2 + \frac{x^4}{1.2} - \frac{x^6}{1.2.3} + \dots \right)$$

$$\therefore \text{area} = \frac{2}{\sqrt{\pi}} \left(x - \frac{1}{1} \frac{x^3}{3} + \frac{1}{1.2} \frac{x^5}{5} - \frac{1}{1.2.3} \frac{x^7}{7} + \dots \right) = \frac{1}{2}$$

$$\therefore x - \frac{1}{1.3} x^3 + \frac{1}{1.2.5} x^5 - \frac{1}{1.2.3.7} x^7 + \frac{1}{1 \frown 4.9} x^9$$

$$- \frac{1}{1 \frown 5.11} x^{11} + \frac{1}{1 \frown 6.13} x^{13} = \frac{\sqrt{\pi}}{4} = .4431134627.$$

From this equation the value of x has to be found: call this equation (A). The following six values of the co-efficients are readily determined:

$$\frac{1}{1.3} = -109866750.$$

$$\frac{1}{1.2.3.4.9} = -537554853.$$

$$\frac{1}{1.2.5} = -230270081.$$

$$\frac{1}{1.2.3.4.5.11} = -718574810.$$

$$\frac{1}{1.2.3.7} = -373785746.$$

$$\frac{1}{1.2.3.4.5.6.13} = -914466007.$$

It is evident that x is greater than .4; then if .4 be substituted for x in the left-hand member of (A), it becomes

$$+ .4000000000 - .0213333333 + .0010240000 - .0000390095 \\ + .0000012136 - .0000000318 + .0000000007.$$

To effect the object in view, only a portion of these terms are required:

$$\begin{array}{r} 400 + \\ 64 - \\ 5 + \\ \hline 341 \end{array}$$

once
3 times
5 times

$$\begin{array}{r} + .40000 \dots \dots \\ - .02133 \dots \dots \\ + .00102 \dots \dots \\ - .00003 \dots \dots \end{array}$$

.37966 take
.44311 from, right mem. of (A).

$$\hline 6345$$

$$\begin{array}{r}
 341 \) \ 6345 \ (\downarrow 1,8, \\
 \underline{341} \\
 2935 \\
 \underline{2728}
 \end{array}$$

So that the first part of the value of

$$x = \frac{4}{10} \downarrow 1,8, = \downarrow - \overline{74141518,}.$$

The succeeding results are obtained by substituting this value of x in the left member of (A) :

$$47645 + \text{once} \quad \cup + \cdot 476\,456\,9505 = x = \frac{4}{10} \downarrow 1,8,$$

$$10816 - 3 \text{ times} \quad \cup - \cdot 036\,053\,6904 = \frac{1}{1.3} x^3$$

$$1228 + 5 \text{ times} \quad \cup + \cdot 002\,455\,3777 = \frac{1}{1.2.5} x^5$$

$$93 - 7 \text{ times} \quad \cup - \cdot 000\,1327139 = \frac{1}{1.2.3.7} x^7$$

$$\frac{5}{37969} + 9 \text{ times} \quad \cup + \cdot 000\,0058581 = \frac{1}{1.2.3.4.9} x^9$$

$$- \cdot 000\,0002175 = \frac{1}{1.2.3.4.5.11} x^{11}$$

$$+ \cdot 000\,0000064 = \frac{1}{1.2.3.4.5.6.13} x^{13}$$

442 | 7315709 take

443 | 1134627 from, the right side of (A).

$$\begin{array}{r}
 37969 \quad | \quad 3818918 \\
 \underline{37969} \quad | \quad 37969 \quad (\downarrow 0,0,1,0,0, \\
 220
 \end{array}$$

Hence, the value of x , as far as the fifth position, will be $\frac{4}{10} \downarrow 1,8,1,0,0,$. Before advancing the next step, it may be

$$\begin{array}{rcl}
 47693 + & \text{once} & + \cdot 476933 \ 4075 \\
 10848 - & 3 \text{ times} & - \cdot 036161 \ 9597 \\
 1234 + & 5 \text{ times} & + \cdot 002467 \ 6792 \\
 93 - & 7 \text{ times} & - \cdot 000133 \ 6457 \\
 5 + & 9 \text{ times} & + \cdot 000005 \ 9110 \\
 \hline
 37991 & & - \cdot 000000 \ 2198 \\
 & & + \cdot 000000 \ 0065
 \end{array}$$

$$\begin{array}{r}
 \cdot 443111 | 1790 \text{ take} \\
 \cdot 443113 | 4627 \text{ from, (A).}
 \end{array}$$

$$\begin{array}{r}
 37991 \quad 2 \overline{) 2837} \quad (\downarrow 0,0,0,0,0,6,0,1,1, \\
 \cdot \cdot \cdot \quad 2 \overline{) 2795} \\
 \hline
 \quad \quad \quad 42 \\
 \quad \quad \quad 38 \\
 \hline
 \quad \quad \quad 4 \\
 \quad \quad \quad 4 \\
 \hline
 \end{array}$$

$$\therefore x = \frac{4}{10} \downarrow 1,8,1,0,0,6,0,1,1,$$

$$\begin{array}{r}
 476933 | 4075 \\
 \cdot \cdot \cdot 2 | 8616 \cup 6, \\
 \quad \quad 48 \cup 0,1, \\
 \quad \quad 5 \cup 0,0,1,
 \end{array}$$

$$x = \cdot 476936 \ 2744$$

Since $y = \frac{2}{\sqrt{\pi}} \epsilon^{-x^2}$, and it was before shown that ϵ reduced to the eight position, or $\downarrow \overline{\epsilon'} = 100005025$; $\downarrow \overline{2'} = 69318201$; $\downarrow \overline{\pi'} = 114478742$, then,

$$-x^2 = -\frac{16}{10^2} \downarrow 2,16,2,0,0,12,0,2,$$

$$\text{and } -100005025 \times \frac{16}{100} \downarrow 2,16,2,0,0,12,0,2, = -22747964$$

$$\sqrt{\pi} = \downarrow \frac{1}{2} \overline{\pi'} = -57239371$$

$$-79987335$$

$$2 \cup = +69318201$$

$$\frac{8}{10} \downarrow 1,2,1,2,4,7,2,5, = -10669134$$

$$\therefore \frac{8}{10} \downarrow 1,2,1,2,4,7,2,5, = \cdot 89880788 = y.$$

The value of e^{-x^2} may be found by two successive operations, thus,

$$100005025 \times \frac{16}{100} = 16000804.$$

$16000804 \downarrow 1,8,1,0,0,6,0,1, = 19059235 \cdot 8$, and
 $19059235 \cdot 8 \downarrow 1,8,1,0,0,6,0,1, = 22747964$, to which append the negative sign.

11. Find the co-ordinates of the curve $y = \frac{2}{\sqrt{\pi}} e^{-x^2}$, when the area is the fraction $\frac{2143}{4789}$; 2143 and 4789 are prime numbers.

From the last problem,

$$\begin{aligned} x - \frac{1}{1.3} x^3 + \frac{1}{1.2.5} x^5 - \frac{1}{1.2.3.7} x^7 + \frac{1}{1.2.3.4.9} x^9 - \dots \\ = \frac{\sqrt{\pi}}{2} \times \frac{2143}{4789} = \cdot 396572205, (A). \end{aligned}$$

$\cdot 4$, may be substituted for x in the left number of (A); then, as in the last example,

400 +	once	+ ·400000 . . .
64 -	3 times	- ·021333 . . .
5 +	7 times	+ ·001024 . . .
341		- ·000039 . . .
		·379652 . . . take
		·396572 . . . from (A).
	341)	16920
		1364
		3280

(↓ 0,4,9,

The first part of the value of x is $\frac{4}{10} \downarrow 0,4,9, = \downarrow - \overline{86753752}$

$$\frac{4}{10} \downarrow 0,4,9, = \cdot 420002800$$

$$- \frac{1}{1.3} x^3 = - \cdot 024696492; \quad - \frac{1}{1.2.3.7} x^7 = - \cdot 000054892;$$

$$+ \frac{1}{1.2.5} x^5 = + \cdot 001306949; \quad + \frac{1}{1.2.3.4.9} x^9 = + \cdot 000001883;$$

$$- \frac{1}{1.2.3.4.5.11} x^{11} = - \cdot 0000000054.$$

12000 +	once	+ ·42000	2800
7409 -	3 times	- ·02469	6492
653 +	5 times	+ ·00130	6956
38 -	7 times	- ·00005	4892
2 +	9 times	+ ·00000	1883
		- ·00000	0054
35208)			
...			

$$\begin{array}{r} \cdot 39656 \mid 0201 \text{ take} \\ \cdot 39657 \mid 2205 \text{ from (A).} \end{array}$$

$$\begin{array}{r} 1 \mid 2004 \\ 1 \mid 0562 \end{array}$$

$$(\downarrow 0,0,0,0,3,4,1,$$

$$\begin{array}{r} 1442 \\ 1408 \end{array}$$

$$4$$

$$35$$

$$\therefore x = \frac{4}{10} \downarrow 0,4,9,0,3,4,1, = \cdot 420017122.$$

$$- x^2 = - \frac{16}{10^2} \downarrow 0,8,18,0,6,8,2, =$$

$$\downarrow, \epsilon^{-x^2} = - 100005025 \times \frac{16}{10^2} \downarrow 0,8,18,0,6,8,2, = - 17652824$$

$$\downarrow, \sqrt{\pi} = \downarrow \frac{1}{2} \pi', \quad = - 57239371$$

$$- 74892195$$

$$\downarrow, 2. = + 69318201$$

$$- 5573994$$

$$\begin{array}{r}
 - \quad 557\,3994 = \frac{9}{10} \downarrow 0,4,9,8,2,6,6, \\
 10 \sim + 23027\,008\,1 \\
 \hline
 22469\,608\,7 \\
 21973\,350\,0 \\
 \hline
 496\,258\,7 \\
 398\,033\,2 \\
 \hline
 98\,225\,5 \\
 899\,595 \\
 \hline
 8,2,6,6,0,
 \end{array}$$

$$\therefore y = \frac{2}{\sqrt{\pi}} \varepsilon^{-x^2} = \frac{9}{10} \downarrow 0,4,9,8,2,6,6, = .945787725.$$

12. Given, $x \sin x = \cos x$, to find x .

It is evident that

$$x \left(x - \frac{x^3}{1 \frown 3} + \frac{x^5}{1 \frown 5} + \dots \right) \text{ must be equal } 1 - \frac{x^2}{2} + \frac{x^4}{1 \frown 4} - \dots$$

$$\therefore \frac{3x^2}{2} - \frac{5x^4}{1 \frown 4} + \frac{7x^6}{1 \frown 6} - \frac{9x^8}{1 \frown 8} + \frac{11x^{10}}{1 \frown 10} - \dots = 1, \quad (1).$$

Put $v = x^2$, then (1) becomes (2),

$$\frac{3v}{2} - \frac{5v^2}{1 \frown 4} + \frac{7v^3}{1 \frown 6} - \frac{9v^4}{1 \frown 8} + \frac{11v^5}{1 \frown 10} - \dots = 1, \quad (2).$$

If $\frac{3v}{2} - \frac{5v^2}{24}$ be put = 1, then the value of v is + 6.4 . . . or + .8 . . . ; therefore .7 cannot be greater than the value of v . Substitute .7 for v in (2), and it becomes

$$\begin{aligned}
 &+ 1.050000000 - .102083333 + .003334722 \\
 &- .000053594 + .000000509
 \end{aligned}$$

$$\begin{array}{r}
 + 1050 \\
 - 204 \\
 + 10 \\
 \hline
 856
 \end{array}$$

once
twice
3 times

$$\begin{array}{r}
 + 1'05|00 \dots\dots \\
 - \cdot 10|20 \dots\dots \\
 + \cdot 00|33 \dots\dots \\
 - \cdot 00|00 \dots\dots \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 \cdot 95|13 \dots\dots \text{take} \\
 1'00|00 \dots\dots \text{from} \\
 \hline
 \end{array}$$

$$856 \) \quad 4|87 \dots\dots \quad (\downarrow 0,5,$$

$$\therefore v = \frac{7}{10} \downarrow 0,5, \dots\dots$$

$$\begin{array}{r}
 \downarrow 0,5, \quad \downarrow 0,10, \quad \downarrow 0,15, \quad \downarrow 0,20, \\
 + 1'050000000 - \cdot 102083333 + \cdot 003334722 - \cdot 000053594 \\
 + 1'103560553 - \cdot 112763508 + \cdot 003871509 - \cdot 000065396
 \end{array}$$

$$\begin{array}{r}
 \downarrow 0,25, \\
 + \cdot 000000509
 \end{array}$$

$$\begin{array}{r}
 + 1103 \\
 - 225 \\
 + 11 \\
 \hline
 889
 \end{array}$$

once
twice
3 times
4 times

$$\begin{array}{r}
 + 1'103|560 \dots \\
 - \cdot 112|763 \dots \\
 + \cdot 003|871 \dots \\
 - \cdot 000|065 \dots \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 \cdot 994|603 \dots \text{take} \\
 1'000|900 \dots \text{from} \\
 \hline
 \end{array}$$

$$889 \) \quad 5|397 \dots \quad (\downarrow 0,0,6,$$

$$\therefore v = \frac{7}{10} \downarrow 0,5,6, \dots\dots$$

$$\begin{array}{r}
 \downarrow 0,5,6, \quad \downarrow 0,10,12, \\
 + 1'050000000 - \cdot 102083333 \\
 + 1'110198495 - \cdot 114124137
 \end{array}$$

$$\begin{array}{r}
 \downarrow 0,15,18, \quad \downarrow 0,20,24, \quad \downarrow 0,25,30, \\
 + \cdot 003334722 - \cdot 000053594 + \cdot 000000509
 \end{array}$$

$$\begin{array}{r}
 \downarrow 1,7,2,2,8,3,4,7 \quad \downarrow 2,3,2,5,2,4,2,7, \quad \downarrow 2,8,8,5,2,4,7,2. \\
 + \cdot 003334722 - \cdot 000053594 + \cdot 000000509 \\
 + \cdot 003941787 - \cdot 000066982 + \cdot 000000673
 \end{array}$$

$$\begin{array}{r}
 + 111019 \\
 - 22824 \\
 + 1182 \\
 - 26 \\
 \hline
 89351
 \end{array}$$

once
twice
3 times
4 times

$$\begin{array}{r}
 + 1.11019 \overline{)8495} \\
 - .11412 \overline{)4137} \\
 + .00394 \overline{)1787} \\
 - .00006 \overline{)6982} \\
 + .00000 \overline{)0673}
 \end{array}$$

$$\begin{array}{r}
 .99994 \overline{)9836} \text{ take} \\
 1.00000 \overline{)0000} \text{ from}
 \end{array}$$

89351)

$$\begin{array}{r}
 5 \overline{)0164} \\
 4 \overline{)4676}
 \end{array}$$

(↓ 0,0,0,0,5,6,1,5,

$$5488$$

$$5361$$

$$127$$

$$89$$

$$38$$

$$\therefore v = \frac{7}{10} \downarrow 0,5,6,0,5,6,1,5,$$

By advancing another step, the value of v may be found correct to twenty places of decimals.

$$v = \frac{7}{10} \downarrow 0,5,6,0,5,6,1,5, = .740173889$$

But $v = x^2$, therefore

$$x = .86033359 = \text{an arc of } 49^\circ 17' 36''.55.$$

The value of v may be found under many forms; for instance:—

$$v = \frac{4}{10} \downarrow 6,4,3,7,5,9,7,4,$$

$$v = \frac{5}{10} \downarrow 4,1,1,0,8,6,5,0,$$

$$v = \frac{6}{10} \downarrow 2,1,9,3,8,9,3,2,$$

$$v = \frac{6}{10} \downarrow 2,2,0,\bar{6},3,4,4,4,$$

each gives $v = .740173889$. This remark applies to all other equations.

GENERAL DEVELOPMENTS.

ANY quantity or magnitude X , capable of being represented by numbers, may be expressed under the form

$$X = 10^m u \downarrow u_1, u_2, u_3, u_4, \dots u_n = \downarrow \overset{n}{\pm} x, \quad (I).$$

u being one of the nine digits 1, 2, 3, 4, 5, 6, 7, 8, or 9.

$u_1, u_2, u_3, \dots$ have the same range of values, positive or negative, and zero. x is a whole number, and may be either positive or negative, and $\downarrow \overset{n}{\pm} x$, represents X , reduced to the n th position. m is a positive or negative whole number. The design and scope of the present work exclude fractional and other combinations: hence all the numbers employed in developing X , are integers, and may be either positive or negative as the case may require. For example, suppose

$$X = 7276.68024, \text{ then,}$$

$$X = 10^3 \times 6 \downarrow 2, 0, 2, 3, \bar{1}, 5, 9, 3, = \downarrow \overset{8}{889287691},$$

Comparing this particular example with the general development (I), $m = 3$; $u = 6$; $u_1 = 2$; $u_2 = 0$; $u_3 = 2$; $u_4 = 3$; $u_5 = \text{minus } 1$; and so on to $u_8 = 3$. In this example $n = 8$, the number of terms employed in the development, and may be extended to 9, 10, . . . to suit any required degree of accuracy. $x = 889287691$, that is, X reduced to the eight position; this ultimate whole number x , may be employed as the logarithm of X ,

and treated accordingly; but it must be observed that this ultimate number x , has many other uses and relations, especially in connexion with $u_1, u_2, u_3, \dots$ and with the function X itself. For instance, suppose that X has to be developed so that the expression may be of the form

$$3 \downarrow 3, 3, 3, 0, 0, 0, 0, x',$$

and yet, to be $= 7276.68024$, what number then will x' represent in the eight position. It is readily shown that

$$x' = \downarrow^8 \overline{747541336},$$

$$\text{or } X = 3 \downarrow 3, 3, 3, 0, 0, 0, 0, \downarrow^8 \overline{747541336}, = \downarrow^8 \overline{889287691},$$

The number of forms under which X may be developed, without changing the value of X , or the final number x , are without limit. Again, take another example, and let $X = .000401428684$.

$$\text{Then } X = \frac{1}{10^4} \times 4 \downarrow 0, 0, 3, 5, 7, \bar{3}, \bar{2}, 7, = \downarrow - \downarrow^8 \overline{782087369},$$

In this state of the function $m = -4$; $n = 4$; $u_1 = 0, u_2 = 0, u_3 = 3, u_4 = 5, u_5 = 7, u_6 = \text{minus } 3, u_7 = 2$, and $u_8 = \text{minus } 1$; hence $n = 8$, and $x = -782087369$.

Let a = the ultimate changeable, but not variable, base

$$1.000 \dots (n-1) \text{ zeros } \dots 1;$$

and the equation $P^z = Q$, is required to be solved.

$$\begin{aligned} P &\text{ may be made to assume the form } 10^m u \downarrow u_1, u_2, u_3, \dots u_n \\ &= \downarrow^{\overline{n}} p, \text{ and } Q \text{ to assume the form } 10^m u \downarrow u_1, u_2, u_3, \dots u_n \\ &= \downarrow^{\overline{n}} q, \end{aligned}$$

$$\therefore (a^p)^z = a^q;$$

then, in any system of logarithms,

$$z \log. (a^p) = q \log. a,$$

$$\text{or, } zp \log. a = q \log. a$$

$$\therefore z = \frac{q}{p}.$$

Also, in any system of logarithms,

$$z \log. P = \log. Q;$$

$$\therefore z = \frac{\log. Q}{\log. P}$$

Hence,
$$\frac{\log. Q}{\log. P} = \frac{q}{p};$$

and, consequently, p and q may be employed as the logarithms, respectively, of P and Q .

Let $a_1, a_2, a_3, \dots$ be the bases of $u_1, u_2, u_3, \dots$ and also of $\bar{u}_1, \bar{u}_2, \bar{u}_3, \dots$; then, because

$$10^m u \downarrow a_1^{u_1} a_2^{u_2} a_3^{u_3} \dots = a^p = P,$$

$$\text{and } 10^m \bar{u} \downarrow a_1^{\bar{u}_1} a_2^{\bar{u}_2} a_3^{\bar{u}_3} \dots = a^q = Q;$$

$$\therefore P \times Q = 10^{m+m} u \bar{u} \downarrow a_1^{u_1 + \bar{u}_1} a_2^{u_2 + \bar{u}_2} a_3^{u_3 + \bar{u}_3} \dots = a^{p+q}$$

$$\therefore \frac{P}{Q} = 10^{m-m} \frac{u}{\bar{u}} \downarrow a_1^{u_1 - \bar{u}_1} a_2^{u_2 - \bar{u}_2} a_3^{u_3 - \bar{u}_3} \dots = a^{p-q}.$$

And further,

$$P^s = 10^{sm} u^s \downarrow a_1^{s u_1} a_2^{s u_2} a_3^{s u_3} \dots = a^{sp}.$$

Consequently, $u_1, u_2, u_3, \dots$ and $\bar{u}_1, \bar{u}_2, \bar{u}_3, \dots$ to the right of $\downarrow$, may be operated upon as logarithms, with regard to their respective bases. The numbers to the left of $\downarrow$, are acted upon by the operations of common arithmetic; but, as these numbers can always be reduced to digits and powers of 10, the operations of common arithmetic to be performed cannot be difficult.

It may be necessary now to repeat, that in this work $\bar{u}$ is put

for du , the differential of u ; ${}^2\overline{u}$ the second differential of u ; ${}^3\overline{u}$ the third differential of u ; and so on. $\overline{u}^2$ is put for du^2 , the square of the differential of u ; $\overline{u}^3$ for the cube of the differential of u , &c. In the same way, ${}^2\overline{u}^3$ represents $(d^2u)^3$, the cube of the second differential of u . Then Maclaurin's, or, more properly speaking, Stirling's, theorem may be expressed as follows:

$$U = \{U\} + \left\{ \frac{\overline{U}}{\overline{x}} \right\} \frac{x}{1} + \left\{ \frac{{}^2\overline{U}}{\overline{x}^2} \right\} \frac{x^2}{1.2} + \left\{ \frac{{}^3\overline{U}}{\overline{x}^3} \right\} \frac{x^3}{1.2.3} + \dots \quad (2).$$

$\{U\}$, $\left\{ \frac{\overline{U}}{\overline{x}} \right\}$, $\left\{ \frac{{}^2\overline{U}}{\overline{x}^2} \right\}$, &c. are intended to represent, by the use of brackets, what U , and the results of the operations expressed by symbols

$$\frac{\overline{U}}{\overline{x}}, \frac{{}^2\overline{U}}{\overline{x}^2}, \frac{{}^3\overline{U}}{\overline{x}^3}, \text{ \&c. become when } x = 0.$$

To develop a^x in a series by (2).

$$\text{Put } U = a^x \quad \text{then } \{U\} = 1$$

$$\frac{\overline{U}}{\overline{x}} = \log. a \cdot a^x \quad \text{then } \left\{ \frac{\overline{U}}{\overline{x}} \right\} = \log. a$$

$$\frac{{}^2\overline{U}}{\overline{x}^2} = (\log. a)^2 a^x \quad \text{then } \left\{ \frac{{}^2\overline{U}}{\overline{x}^2} \right\} = (\log. a)^2$$

$$\frac{{}^3\overline{U}}{\overline{x}^3} = (\log. a)^3 a^x \quad \text{then } \left\{ \frac{{}^3\overline{U}}{\overline{x}^3} \right\} = (\log. a)^3$$

$$\text{\&c.} = \text{\&c.} \quad \text{then } \text{\&c.} = \text{\&c.}$$

$$\therefore a^x = 1 + (\log. a) \frac{x}{1} + (\log. a)^2 \frac{x^2}{1.2} + (\log. a)^3 \frac{x^3}{1.2.3} \dots \quad (3).$$

When $x = 1$, (3) becomes (4),

$$a = 1 + \frac{(\log. a)}{1} + \frac{(\log. a)^2}{1.2} + \frac{(\log. a)^3}{1.2.3} + \dots \quad (4).$$

(4) being inverted gives (5),

$$\log. a = (a - 1) - \frac{(a - 1)^2}{2} + \frac{(a - 1)^3}{3} - \dots \quad (5).$$

Since $a = 1.000 \dots (n - 1) \text{ zeros} \dots 1$

$$\therefore \log. a = .000 \dots (n - 1) \text{ zeros} \dots 1 = (a - 1),$$

as the remaining part of (5) after $(a - 1)$, has no influence within the proposed degree of accuracy.

For the purpose of illustration, let $a^x = X$, then,

$$X = 1.34985882 = \downarrow 3,1,4,1,2,1,1,3, = \downarrow \overset{8}{\text{30000000}},$$

which being compared with (3), gives $x = 30000000$,

$$n = 8, a = 1.000000001, \log. a = .000000001.$$

K	L
10000000	$\dots\dots\dots = 1$
3000000	$\dots\dots\dots = \frac{x}{1} \log. a = B$
450000	$\dots\dots\dots = \frac{x-1}{2} B \log. a = \frac{x^2}{1.2} (\log. a)^2 = C$
45000	$\dots\dots\dots = \frac{x-2}{3} C \log. a = \frac{x^3}{1.2.3} (\log. a)^3 = D$
3375	$\dots\dots\dots = \frac{x-3}{4} D \log. a = \frac{x^4}{1.2.3.4} (\log. a)^4 = E$
203	$\dots\dots\dots = \frac{x-4}{5} E \log. a = \frac{x^5}{1.2.3.4.5} (\log. a)^5 = F$
10	$\dots\dots\dots = \frac{x-5}{6} F \log. a - \frac{x^6}{1.2.3.4.5.6} (\log. a)^6 = G$
(6). 13498588	

$\frac{x-6}{7} G \log. a$ will not increase the period K a unit, the result would fall under L ; the values of $C, D, E, \dots$ under K , are not affected by putting x for $(x - 1)$, or for $(x - 2)$, &c. Take the term E , for example :

$$\begin{array}{rcl}
 x - 3 & = & 29999997 \text{ Mult.} \\
 D & = & 45000 \text{ by} \\
 4 \overline{) 13499\,99865000} & & \\
 \log. a & = & \frac{3374\,99966250}{\cdot 000\,00001} \text{ Mult.} \\
 & & \text{by} \\
 \frac{x-3}{4} D \log. a & = & 3374\,99966250 = 3375.
 \end{array}$$

$$\begin{array}{rcl}
 x & = & 300\,00000 \text{ Mult.} \\
 D & = & 45000 \text{ by} \\
 4 \overline{) 13500\,00000000} & & \\
 \log. a & = & \frac{3375\,00000000}{\cdot 00\,000001} \text{ Mult.} \\
 & & \text{by} \\
 \frac{x}{4} D \log. a & = & 3375\,00000000.
 \end{array}$$

Hence, the result E , within the limits of the required range of accuracy, is not affected by putting $x - 3$ for x , and it is evident that the range of accuracy may be extended to any given limit.

$$\begin{array}{r}
 1 \overline{) 00000000} \\
 3 \overline{) 00000000} \\
 3 \overline{) 00000000} \\
 1 \overline{) 00000000}
 \end{array}$$

$$\begin{array}{r}
 13 \overline{) 31000000} \\
 13 \overline{) 310000}
 \end{array}$$

$$\begin{array}{r}
 134 \overline{) 431000} \\
 537 \overline{) 724} \\
 807
 \end{array}$$

$$\begin{array}{r}
 1349 \overline{) 69532} \dots \\
 1349 \overline{) 7} \dots
 \end{array}$$

$$\begin{array}{r}
 134983028 \\
 \dots 2700 \\
 135 \\
 13 \\
 4
 \end{array}$$

$$(7). \quad 134985881 = \downarrow 3, 1, 4, 1, 2, 1, 1, 3,$$

$\therefore$ (7) corresponds exactly with (6).

The equality between $\downarrow 3,1,4,1,2,1,1,3$, and $\downarrow \overline{8}30000000$, remains to be established. It will be shown presently, that the ultimate values of $\downarrow 1$, ; $\downarrow 0,1$, ; and $\downarrow 0,0,1$, in the eight position, will be

$$\downarrow 1, = \downarrow \overline{8}9531018,$$

$$\downarrow 0,1, = \downarrow \overline{8}995033,$$

$$\downarrow 0,0,1, = \downarrow \overline{8}99950,$$

$$3 \times 9531018 = 28593054$$

$$1 \times 995033 = 995033$$

$$4 \times 99950 = 399800$$

$$\text{and} \quad \underline{1,2,1,1,3,}$$

$$30000000.$$

$$\therefore \downarrow 3,1,4,1,2,1,1,3, = \downarrow \overline{8}30000000,$$

and the connexion shown to exist generally is established in this particular case.

ULTIMATE VALUES

of $\downarrow 1$, $\downarrow 0,1$, $\downarrow 0,0,1$, &c. and of 2, 3, 4, &c. in the eight position.

Referring to the equalities (page 42), and the method of calculation employed (pages 52 and 53), the succeeding deductions are readily drawn.

$$\downarrow 0,0,0,1, = 9999.54 \text{ (d).}$$

$$\text{Then, } \downarrow 0,0,0,9, = (d) \times 9 = 89995.86$$

$$\text{and } \downarrow 0,0,0,0,9,9,5,4,5,7, = 9954.57$$

$$\therefore \downarrow 0,0,1, = \downarrow 0,0,0,9,9,9,5,4,5,7, = 99950.43 \text{ (c).}$$

$$\text{Then, } \downarrow 0,0,9, = (c) \times 9 = 899550.$$

$$\text{and } \downarrow 0,0,0,9, = (d) \times 9 = 89995.86$$

$$\text{and } \downarrow 0,0,0,0,5,4,8,7,3,1, = 5487.31$$

$$\downarrow 0,1, = \downarrow 0,0,9,9,5,4,8,7,3,1, = 995033.17 \text{ (b).}$$

$$\text{Then, } \downarrow 0,9, = (b) \times 9 = 8955297$$

$$\text{and } \downarrow 0,0,5, = (c) \times 5 = 499752$$

$$\text{and } \downarrow 0,0,0,7, = (d) \times 7 = 69996$$

$$\text{and } \downarrow 0,0,0,0,5,9,7,3, = 5973$$

$$\therefore \downarrow 1, = \downarrow 0,9,5,7,5,9,7,3, = 9531018 \text{ (a).}$$

Therefore the ultimate values of $\downarrow 1$, $\downarrow 0,1$, $\downarrow 0,0,1$, and $\downarrow 0,0,0,1$, in the eight position becomes known, and are respectively 9531018 (a), 995033 (b), 99950 (c), and 10000 (d.) Ultimate values of $\downarrow 1$, $\downarrow 0,1$, &c. in any other position may be found in a similar manner. As multiples of these ultimate values of $\downarrow 1$, $\downarrow 0,1$, $\downarrow 0,0,1$, &c. in the eight position may be found useful, they are here appended.

	$\downarrow 1, \text{ (a).}$	$\downarrow 0,1, \text{ (b).}$	$\downarrow 0,0,1, \text{ (c).}$	$\downarrow 0,0,0,1, \text{ (d).}$
1	9531018	995033	99950	10000
2	19062036	1990066	199900	20000
3	28593054	2985099	299850	29999
4	38124072	3980132	399800	39998
5	47655090	4975165	499750	49998
6	57186108	5970198	599702	59997
7	66717126	6965231	699652	69997
8	76248144	7960264	799602	79996
9	85779162	8955297	899552	89996

It will be found, on referring to page 54, that

$$2^{\cdot} = \downarrow 7, 2, 6, 0, 7, 8, 2, 6,$$

$$\text{Then, } \downarrow 7, \quad = 66717126$$

$$2, \quad = 1990066$$

$$6, \quad = 599700$$

$$\text{and} \quad 7826$$

$$\left. \begin{array}{l} \text{The ultimate value of 2,} \\ \text{in the eight position} \end{array} \right\} = 69314718$$

$$\text{Again, } 3^{\cdot} = \downarrow 11, 5, 0, 4, 4, 8, 6, 8,$$

$$\text{Then, } \downarrow 10, \quad = 95310180$$

$$\downarrow 1, \quad = 9531018$$

$$5, \quad = 4975165$$

$$0, 4, \quad = 39998$$

$$\text{and} \quad 4868$$

$$\left. \begin{array}{l} \text{The ultimate value of 3,} \\ \text{in the eight position} \end{array} \right\} = 109861229$$

In a similar manner, the ultimate values of 4, 5, 6, &c. in the eight, or in any other position, may be found and registered for use.

$$2^{\cdot} = \downarrow \overset{8}{\overline{69314718}},$$

$$7^{\cdot} = \downarrow \overset{8}{\overline{194591016}},$$

$$3^{\cdot} = \downarrow \overset{8}{\overline{109861229}},$$

$$8^{\cdot} = \downarrow \overset{8}{\overline{207944155}},$$

$$4^{\cdot} = \downarrow \overset{8}{\overline{138629437}},$$

$$9^{\cdot} = \downarrow \overset{8}{\overline{219722459}},$$

$$5^{\cdot} = \downarrow \overset{8}{\overline{160943792}},$$

$$10^{\cdot} = \downarrow \overset{8}{\overline{230258510}},$$

$$6^{\cdot} = \downarrow \overset{8}{\overline{179175948}},$$

$$11^{\cdot} = \downarrow \overset{8}{\overline{239789528}},$$

These ultimate values will often be found useful; for example, let it be required to show that

$$\log. \pi = 1.6\frac{5}{9} + \log. \frac{2}{3},$$

within the limits of the eight position

$$\frac{\pi}{6} = .5235987755$$

$$\therefore \frac{10\pi}{6} = \frac{5\pi}{3} = 5 \downarrow 0,4,6,3,1,9,2,9,$$

Then taking the ultimate values of these quantities,

$$\begin{array}{rcl} & 160943792 = 5. & \\ \downarrow 0,4, & = & 3980132 \\ 6, & = & 599702 \\ \text{and} & & 31929 \\ \hline & & 165555555 \end{array}$$

$$\therefore \frac{5\pi}{3} \text{ reduced to the eight position becomes } \downarrow \overset{8}{165555555},$$

$$\epsilon = \downarrow 10,4,7,1,0,0,3,8, \text{ (see page 123).}$$

$$\begin{array}{rcl} \text{Then } \downarrow 10, & = & 95310180 \\ & 4, & = 3980132 \\ & 7, & = 699652 \\ & \text{and,} & 10038 \\ \hline & & 100000002 \end{array}$$

$$\therefore \frac{165555555}{100000000} = 1.6\frac{5}{9}$$

$$\therefore \log. \frac{5\pi}{3} = 1.6\frac{5}{9}$$

$$\text{or } \log. \pi + \log. 5 - \log. 3 = 1.6\frac{5}{9}$$

$$\therefore \log. \pi = 1.6\frac{5}{9} + \log. \frac{2}{3},$$

a result easily verified to be correct as far as the eight position.

$$\begin{array}{r}
 \text{Log. } 3 = \frac{1.6555555}{1.0986123} \\
 \log. 5 = \frac{2.7541678}{1.6094379} \\
 1.1447299 = \text{Hyp. log. } \pi.
 \end{array}$$

Magnitudes may be presented under the form

$$2^r \{ \dots w_3, w_2, w_1, \downarrow u_1, u_2, u_3, \dots \},$$

in which operative figures stand on the right and left of the sign $\downarrow$, 2^r has a common arithmetical interpretation; but such developments are designedly omitted in the present work. However, to illustrate this matter it is easily shown that

161942.257612 may be represented by

$$\{0, 5, \downarrow 0, 5, 5, 2, 1, 2, 7, 5, ; 30027836.8 \text{ by}$$

$$2 \{0, 2, 3, \downarrow 1, 0, 5, 2, 5, 3, 6, 1, \text{ and}$$

$$5341.18272 \text{ by } 2^2 \{0, 3, \downarrow 0, 0, 3, 2, 2, 3, 7, 3,$$

NOTES TO THE INTRODUCTORY EXAMPLES.

OPERATIONS OMITTED
IN PASSING FROM STEP TO STEP IN THE
INTRODUCTORY EXAMPLES.

NOTES

TO THE

INTRODUCTORY EXAMPLES.

1911-12
1912-13
1913-14
1914-15
1915-16

1916-17 1917-18 1918-19

1919-20 1920-21 1921-22

1922-23 1923-24 1924-25

1925-26 1926-27 1927-28 1928-29 1929-30
1930-31 1931-32 1932-33 1933-34 1934-35
1935-36 1936-37 1937-38 1938-39 1939-40
1940-41 1941-42 1942-43 1943-44 1944-45

NOTES

1945-46 1946-47 1947-48 1948-49 1949-50

1950-51 1951-52 1952-53 1953-54 1954-55

1955-56 1956-57 1957-58 1958-59 1959-60

1960-61 1961-62 1962-63 1963-64 1964-65

INTRODUCTION EXAMPLER

NOTES TO THE INTRODUCTORY EXAMPLES.

OPERATIONS OMITTED IN PASSING FROM STEP TO STEP IN THE INTRODUCTORY EXAMPLES.

Example I.

$$2 \mid 2092118\cdot5 = 10^6 \times 2 \downarrow 0,4,5,2,3,1,2,0,$$

diff.	104605925
	10 00 00 00 00
	40 00 00 00
	60 00 00
	40 00
	10
	104 060 401
	520 302
	1041
	1
1045 81745...	
20916...	
1...	
10460 2662	
3263	
3138	
125	
105	
20	
20	
—	

$10^6 =$	1381620486
$2 =$	69318201
$\downarrow 0,4, =$	3980332
$5 =$	499775
and	23120
	1455441914

$$\therefore \downarrow 1455441914, \text{ in the eight position, } = 2092118\cdot5.$$

$$7 \mid 78539 \cdot 816 = 10^4 \times 7 \downarrow 1,1,9,8,5,4,4,8,$$

$$\begin{array}{r} \text{diff.} \left\{ \begin{array}{l} \overline{112199737} \\ \begin{array}{c} 1 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \\ \quad 1 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \end{array} \\ \hline 1 \mid 1 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \\ \quad \quad 1 \mid 1 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \\ \hline 1 \mid 1 \mid 1 \mid 1 \mid 0 \mid 0 \mid 0 \mid 0 \\ \quad \quad \quad 9 \mid 9 \mid 9 \mid 9 \mid 0 \mid 0 \\ \quad \quad \quad \quad 4 \mid 0 \mid 0 \mid 0 \\ \quad \quad \quad \quad \quad 9 \\ \hline 1 \mid 1 \mid 2 \mid 1 \mid 0 \mid 3 \mid 9 \mid 0 \dots \\ \quad \quad \quad \quad \quad 8 \mid 9 \mid 6 \mid 8 \mid 3 \dots \\ \quad \quad \quad \quad \quad \quad 3 \mid 2 \dots \\ \hline 1 \mid 1 \mid 2 \mid 1 \mid 9 \mid 3 \mid 6 \mid 2 \mid 4 \\ \quad \quad \quad \quad \quad \quad \quad 6 \mid 1 \mid 1 \mid 3 \\ \quad \quad \quad \quad \quad \quad \quad \quad 5 \mid 6 \mid 1 \mid 0 \\ \hline \quad \quad \quad \quad \quad \quad \quad \quad 5 \mid 0 \mid 3 \\ \quad \quad \quad \quad \quad \quad \quad \quad \quad 4 \mid 4 \mid 9 \\ \hline \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 5 \mid 4 \\ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 4 \mid 5 \\ \hline \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 9 \\ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 9 \\ \hline \end{array} \right. \end{array}$$

$$\begin{array}{rcl} \downarrow 1, & = & 9531497 \\ 1, & = & 995083 \\ 9, & = & 899595 \\ \text{and,} & & 85448 \end{array}$$

$$\begin{array}{rcl} & & 11511623 \\ 7 & = & 194600794 \\ 10^4 & = & 921080324 \\ \hline & & 1127192741 \end{array}$$

$\therefore$ In the eight position, $78539 \cdot 816$ is represented by $\downarrow 1127192741,$

Examples II. and III.

In Example III. it was stated that

$$1 \cdot 2500000000 = \downarrow 2,3,2,6,7,3,2,3, = \downarrow 22315476,1.$$

$$\begin{array}{r} 1 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \\ \quad 2 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \\ \quad \quad 1 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \\ \hline 1 \mid 2 \mid 1 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \end{array}$$

$$\begin{array}{r|l}
 12 & 100000 \\
 & 363000 \\
 & 363000 \\
 & 1210
 \end{array}$$

$$\begin{array}{r|l}
 124 & 6664210 \dots \\
 & 2493328 \dots \\
 & 1247 \dots
 \end{array}$$

$$\begin{array}{r|l}
 1249 & 158785 \dots \\
 & 749495 \dots \\
 & 187 \dots
 \end{array}$$

$$\begin{array}{r|l}
 12499 & 08467 \\
 & 87494 \dots \\
 & 3750 \dots \\
 & 250 \dots \\
 & 37 \dots
 \end{array}$$

$$1249999998$$

The following equations of equality reduced to the eight position are established in the early part of the work.

$$\downarrow 1, = \downarrow 0, 10, \bar{4}, \bar{1}, \bar{9}, \bar{5}, \bar{1}, \bar{3}, = \downarrow \overline{9531497},$$

$$\downarrow 0, 1, = \downarrow 0, 0, 10, 0, \bar{4}, \bar{4}, \bar{6}, \bar{7}, = \downarrow \overline{995083},$$

$$\downarrow 0, 0, 1, = \downarrow 0, 0, 0, 10, 0, 0, \bar{4}, \bar{5}, = \downarrow \overline{99955},$$

$$\downarrow 0, 0, 0, 1, = \downarrow 0, 0, 0, 0, 10, 0, 0, 0, = \downarrow \overline{10000},$$

$$\&c. = \&c. = \&c.$$

$$\downarrow 0, 0, 0, 10, 0, 0, \bar{4}, \bar{5}, = 100000 - 45 = \overline{99955},$$

$$\text{ten times } 99955 = 999550$$

4467 minus

$$\overline{995083},$$

$$\text{ten times } 995083 = 9950830$$

$$\text{minus four times } 99955 = \underline{399820}$$

$$\text{minus } \begin{array}{r} 9551010 \\ 19513 \end{array}$$

$$\overline{9531497},$$

$$\begin{array}{rcl}
 2 \text{ times } 9531497 & = & 19062994 \\
 3 \text{ „ } 995083 & = & 2985249 \\
 2 \text{ „ } 99955 & = & 199910 \\
 & & 67323
 \end{array}$$

$$\therefore \downarrow 2,3,2,6,7,3,2,3, = \downarrow \overline{22315476,}$$

The values of 2, 3, 4, &c. reduced to the eight or any other position, involving whole numbers only, are easily found as follows :—

$$2^{\cdot} = \downarrow 7, 2,6,0,7,8,2,6, = \downarrow \overline{69318201,}$$

$$3^{\cdot} = \downarrow 1,1,5,0,4,4,8,6,8, = \downarrow \overline{109866750,}$$

$$4^{\cdot} = \downarrow 14,5,2,2,0,1,1,9, = \downarrow \overline{138636402,}$$

$$5^{\cdot} = \downarrow 16,8,4,8,7,4,4,3, = \downarrow \overline{160951879,}$$

$$6^{\cdot} = \downarrow 18,7,6,5,2,6,9,4, = \downarrow \overline{179184951,}$$

$$7^{\cdot} = \downarrow 20,3,9,8,6,0,1,0, = \downarrow \overline{194600794,}$$

$$8^{\cdot} = \downarrow 21,7,8,2,7,9,4,6, = \downarrow \overline{207954604,}$$

$$9^{\cdot} = \downarrow 23,0,5,0,9,2,9,4, = \downarrow \overline{219733500,}$$

$$10^{\cdot} = \downarrow 24,1,5,1,9,2,9,5, = \downarrow \overline{230270081,}$$

By a simple problem in the early part of the work, it is easily shown that $\cdot 2 = \downarrow 7,2,6,0,7,8,2,6$, which is easily reduced to $\downarrow \overline{69318201,}$

$$2^2 = 4^{\cdot} \text{ and } 2^3 = 8^{\cdot}$$

$$69318201 \times 2 = 138636402$$

$$69318201 \times 3 = 207954604$$

$$8 \times 1\cdot 25 = 10^{\cdot}$$

But it has been just shown that $1\cdot 25 = \downarrow \overline{22315476,}$

$$\therefore 207954604 + 22315476 = 230270080.$$

Again, $10 \div 9 = 1.11111111 = \downarrow 1,1,0,1,0,0,0,1, = \downarrow \overline{10536581},$

$$\begin{array}{rcl} \downarrow 1, & = & 9531497 \\ \downarrow 0,1, & = & 995083 \\ \text{and} & & 10001 \\ \hline & 10536581 & \text{take} \\ \text{for } 10. & 230270081 & \text{from} \\ \hline \text{for } 9. & 219733500 & \\ \text{for } 3. & 109866750 & \text{half} \\ \hline \end{array}$$

$$\begin{array}{rcl} \text{for } 2. & 69318201 & \\ \text{for } 3. & 109866750 & \\ \hline & 179184951 & \text{for } 6. \end{array}$$

$$\begin{array}{rcl} \text{for } 10. & 230270080 & \\ \text{for } 2. & 69318201 & \\ \hline & 160951879 & \text{for } 5. \end{array}$$

All the numbers are now established except the number for 7.

$$\frac{7}{8} = 1.166666667 = \downarrow 1,5,9,0,9,3,3,5,$$

$$\begin{array}{r} \text{diff.} \left\{ \begin{array}{l} \overline{11000000} \\ 55000000 \\ 11000000 \\ 110000 \\ 55 \end{array} \right. \\ \hline \begin{array}{r} 1156111055 \dots \\ 10404999 \dots \\ 416209 \dots \\ 97 \dots \end{array} \\ \hline 1166557771 \\ \dots \dots \dots \\ \begin{array}{r} 108896 \\ 104989 \\ \hline 3907 \end{array} \end{array}$$

$$\begin{array}{rcl} \downarrow 1, & = & 9531497 \\ 0,5, & = & 4975415 \\ 0,0,9, & = & 899595 \\ \text{and} & & 9335 \\ \hline & 15415842 & \\ 6. = & 179184951 & \\ \hline & 194600793 & \end{array}$$

$$\begin{array}{r}
 3907 \\
 3500 \\
 \hline
 407 \\
 350 \\
 \hline
 57 \\
 58 \\
 \hline
 \end{array}$$

$$\therefore 7 \times \downarrow 194600793, = \downarrow 20,3,9,8,6,0,1,0,$$

In example 2 it is stated that

$$\epsilon = 2.718281828 = \downarrow 100005025,$$

$$\begin{array}{l}
 \frac{\epsilon}{2} = \left[\begin{array}{l}
 \text{diff.} \left\{ \begin{array}{l}
 1.359140914 = \downarrow 3,2,1,0,2,2,1,2, \\
 \begin{array}{|c|c|c|c|c|}
 \hline
 13 & 31 & 00 & 00 & 00 \\
 \hline
 & 26 & 62 & 00 & 00 \\
 \hline
 & & 13 & 31 & 00 \\
 \hline
 \end{array} \\
 \begin{array}{|c|c|c|c|c|}
 \hline
 135 & 775 & 310 & 00 & \dots \\
 \hline
 & 135 & 775 & 3 & \dots \\
 \hline
 \end{array} \\
 \begin{array}{|c|c|c|c|c|}
 \hline
 13591 & 10853 & & & \\
 \hline
 & 30061 & & & \\
 \hline
 & 27182 & & & \\
 \hline
 & 2879 & & & \\
 & 2718 & & & \\
 \hline
 & 161 & & & \\
 & 136 & & & \\
 \hline
 & 25 & & & \\
 & 27 & & & \\
 \hline
 \end{array}
 \end{array} \right.
 \end{array}
 \end{array}$$

$$\begin{array}{rcl}
 \downarrow 3 & = & 28594491 \\
 0,2, & = & 1990166 \\
 0,0,1, & = & 99955 \\
 \text{and} & & 2212 \\
 \hline
 & & 30686824 \\
 2. & = & 69318201 \\
 \hline
 & & 100005025
 \end{array}$$

$$\therefore \epsilon = \downarrow 3,2,1,0,2,2,1,2, = \downarrow 100005025,$$

In the second example it is also stated that

$$\pi = 3.141592654 = \downarrow 114478742,.$$

$$\frac{\pi}{3} = \begin{array}{l} \text{diff.} \left\{ \begin{array}{l} \overline{1'047197551} = \downarrow 0,4,6,3,1,9,2,9, \\ \begin{array}{r} 104 \overline{0604010} \dots \\ \quad 624 \overline{3624} \dots \\ \quad \quad 1 \overline{5609} \dots \\ \quad \quad \quad 21 \dots \end{array} \\ \hline \begin{array}{r} 1046 \overline{863264} \dots \\ \quad 3140 \overline{59} \dots \\ \quad \quad 31 \dots \end{array} \\ \hline \begin{array}{r} 10471 \overline{77354} \\ \dots \overline{20197} \\ \quad 10472 \\ \hline 9725 \\ 9424 \\ \hline 301 \\ 209 \\ \hline 92 \\ 94 \end{array} \end{array} \right. \end{array}$$

$$\begin{array}{l} \downarrow 0,4, = 3980332 \\ 6, = 599730 \\ \text{and} \quad 31929 \\ \hline 4611991 \\ 3' = 109866750 \\ \hline 114478741 \end{array}$$

$$\therefore \pi = \downarrow \overline{114478741} = \downarrow 12,0,1,0,0,8,2,3,$$

Example IV.

It was assumed that $497 \times 23 = 11431$, because

$$\begin{array}{r} 23 \overline{) 114478742} \\ \underline{92} \\ 224 \quad (467 \\ \underline{207} \\ 177 \\ \underline{161} \\ G G \end{array}$$

Example V.

$$10596085\cdot33 = \downarrow 0,5,8,1,5,1,8,8,$$

$$\begin{array}{r} 1000 \mid 00 \mid 00 \mid 00 \\ 50 \mid 00 \mid 00 \mid 00 \\ 10 \mid 00 \mid 00 \mid 00 \\ 100 \mid 00 \\ 50 \end{array}$$

$$\begin{array}{r} 105 \mid 101 \mid 005 \mid 0 \dots \\ 84 \mid 0808 \mid 0 \dots \\ 294 \mid 28 \dots \\ 59 \dots \end{array}$$

$$\begin{array}{r} 1059447617 \dots \\ 105945 \dots \end{array}$$

$$\begin{array}{r} 10595 \parallel 53562 \\ \dots \parallel 54971 \\ 52978 \end{array}$$

$$\begin{array}{r} 1993 \\ 1060 \end{array}$$

$$\begin{array}{r} 933 \\ 848 \end{array}$$

$$\begin{array}{r} 85 \\ 85 \end{array}$$

$$\begin{array}{r} \downarrow 0,5, = 4975415 \\ 0,0,8, = 799640 \\ \text{and} \quad 15188 \\ \text{for } 9 \cdot \quad 219733500 \\ \hline 225523733 \end{array}$$

$$98 \times 23 \quad 2255237430 = \downarrow 0,0,0,5,5,1,1,2,$$

$$\begin{array}{r} 2254 \mid 00 \mid 00 \mid 00 \dots \\ 11 \mid 270 \mid 00 \dots \\ 225 \dots \end{array}$$

$$\begin{array}{r} 2255 \mid 1 \parallel 27225 \\ 1 \parallel 12756 \\ 2 \end{array}$$

$$\begin{array}{r} 225523 \parallel 9983 \\ \dots \parallel 2553 \cdot \\ 2255 \\ \hline 298 \end{array}$$

$$\begin{array}{r} 298 \\ 226 \\ \hline 72 \\ 68 \\ \hline \end{array}$$

$$\downarrow 0,0\bar{1},4,\bar{2},\bar{5},\bar{1},\bar{1}\bar{1}, = \downarrow 0,0,\bar{1},4,\bar{2},\bar{5},\bar{2},\bar{1}, \\ = \downarrow 0,0,\bar{1},3,7,4,7,9,$$

Because $\downarrow 0,0,0,4,0,0,0,0, = \downarrow \overline{40000},$

and $\downarrow 0,0,0,0,2,5,2,1, = \downarrow \overline{2521},$

Example VI.

$$\pi : 180 \times 60 \times 60 :: \text{length} : \text{seconds} = \frac{180 \times 60 \times 60 \times \text{length}}{\pi};$$

but $200000 \pi \downarrow 0,3,1,0,0,\bar{7},0,\bar{5}, = 180 \times 60 \times 60,$

$$\therefore \text{seconds} = \text{length} \times 200000 \downarrow 0,3,1,0,0,\bar{7},0,\bar{5},$$

Example VII.

The length of an arc of $1'' = \cdot 000004848136811;$

$$\cdot 4848136811 \div \cdot 48 = \cdot 10100285 = \downarrow 0,1,0,0,2,8,2,2,$$

To multiply a given number, as 3457 by 48:—

$$\begin{array}{r} 2) 345700 \text{ 100 times} \\ \hline 172850 \text{ 50 times} \\ 6914 \text{ twice} \\ \hline 165936 \text{ 48 times.} \end{array}$$

Hence the truth of these rules is established.

Example VIII.

It is evident that $\downarrow \overline{16265543}$, which is 7 times $\downarrow \overline{2323649}$, may be written $\downarrow 1,6,7,6,3,8,6,3$, because

$$\begin{array}{r} \text{Constant (A), } 9531497 \text{) } \overline{16265543} \\ \quad \quad \quad 9531497 \text{ (}\downarrow 1, \\ \text{Constant (B), } 995083 \text{) } \overline{6734046} \\ \quad \quad \quad 5970498 \text{ (}\downarrow 0,6, \\ \text{Constant (C), } 99955 \text{) } \overline{763548} \\ \quad \quad \quad 699685 \text{ (}\downarrow 0,0,7, \\ \quad \quad \quad \hline \quad \quad \quad 63863 = \downarrow 0,0,0,6,3,8,6,3, \end{array}$$

SECOND SOLUTION.

8. Given the obliquity of the ecliptic = $23^\circ 27' 25''.42$ = $84445''.42$, find the natural sine and log. sine of this angle.

$$\begin{array}{r} 84445''.4200 \\ 42222 \ 7100 \\ 1688 \ 9084 \\ \hline \end{array}$$

$$\text{Length of arc} = .40533 \ 8016 \downarrow 0,1,0,0,2,8,2,2, = .409402949.$$

$$.409402949 = .4 \downarrow 0,2,3,3,3,6,1,8, = .4 \downarrow \overline{2323649},$$

$$\begin{array}{l} \text{cube} = .064 \downarrow \overline{6970947}, \\ = .064 \downarrow 0,7,0,0,5,3,6,6, \end{array}$$

$$\begin{array}{l} \text{fifth power} = .01024 \downarrow \overline{11618245}, \\ = .01024 \downarrow 1,2,0,9,6,5,8,2, \end{array}$$

$$\begin{array}{l} \text{seventh position} = .0016384 \downarrow \overline{16265543}, \\ = .0016384 \downarrow 1,6,7,6,3,8,6,3, \end{array}$$

$$\div 6 = .010666667$$

$$\div 120 = .000085333$$

$$\div 5040 = .000000325$$

$$10666667 \downarrow 0,7,0,0,5,3,6,6 = \begin{array}{r} .409402949 + \\ 11436725 - \end{array}$$

$$85333 \downarrow 1,2,0,9,6, \&c. = \begin{array}{r} 397966224 \\ 95845 + \end{array}$$

$$325 \downarrow 1,6,7,6, \&c. = \begin{array}{r} 398062069 \\ 383 \end{array}$$

$$\text{Natural sine of } 23^\circ 27' 25''.42 = \underline{\underline{.398061686}}$$

$$3) \underline{3.98061686}$$

$$\underline{1.32687229} = \downarrow 2,9,2,6,5,2,2,1, = \downarrow \underline{28283872,}$$

$$(\text{Example 3,}) \quad 3. = \downarrow \underline{109866750,}$$

$$\therefore 3.98061686 = \underline{138150622,}$$

Then 138150622 divided by the constant 230270081 gives .59995038.

$$\therefore \text{Log. sin } 23^\circ 27' 25''.42 = 9.59995038$$

$$2 = \begin{array}{r} 69318201 \\ 109866750 \end{array}$$

$$\begin{array}{r} 179184951 \\ 138636402 \end{array} = 2.3.$$

$$\begin{array}{r} 317821353 \\ 160951879 \end{array} = 2.3.4.$$

$$\begin{array}{r} 478773232 \\ 179184951 \end{array} = 2.3.4.5.$$

$$\begin{array}{r} 657958183 \\ 194600795 \end{array} = 2.3.4.5.6.$$

$$\begin{array}{r} 852558978 \\ 207954604 \end{array} = 2.3.4.5.6.7.$$

$$\begin{array}{r} 1060513582 \\ 219733500 \end{array} = 2.3.4.5.6.7.8.$$

$$\begin{array}{r} 1280247082 \\ 230270081 \end{array} = 2.3.4.5.6.7.8.9.$$

$$\underline{1510517163}$$

$$\begin{array}{r} 1510517163 \\ 239801578 \\ \hline \end{array} = 2.3.4.5.6.7.8.9.10.$$

$$\begin{array}{r} 1750318741 \\ 248503152 \\ \hline \end{array} = 2.3.4.5.6.7.8.9.10.11.$$

$$1998821893 = 2.3.4.5.6.7.8.9.10.11.12.$$

Which may be continued at pleasure by simple addition.

Example IX.

The following simple additions and subtractions may be made before commencing to operate :

$\begin{array}{r} 69318201 \\ 138636402 \\ \hline \end{array}$	$\begin{array}{r} 109866750 \\ 160951879 \\ \hline \end{array}$
$2.4 = \begin{array}{r} 207954603 \\ 179184951 \\ \hline \end{array}$	$3.5 = \begin{array}{r} 270818629 \\ 194600795 \\ \hline \end{array}$
$2.4.6 = \begin{array}{r} 387139554 \\ 207954604 \\ \hline \end{array}$	$3.5.7 = \begin{array}{r} 465419424 \\ 219733500 \\ \hline \end{array}$
$2.4.6.8 = \begin{array}{r} 595094158 \\ 230270081 \\ \hline \end{array}$	$3.5.7.9 = \begin{array}{r} 685152924 \\ 239801578 \\ \hline \end{array}$
$2.4.6.8.10 = \begin{array}{r} 825364239 \\ 248503152 \\ \hline \end{array}$	$3.5.7.9.11 = 924954502$
$2.4.6.8.10.12 = 1073867391$	

$$\frac{1}{2.3} = 179184951 \text{ negative}$$

$$\frac{1.3}{2.4.5} = 259039732 \text{ negative}$$

$$\frac{1.3.5}{2.4.6.7} = 310921720 \text{ negative}$$

$$\frac{1.3.5.7}{2.4.6.8.9} = 349408234 \text{ negative}$$

$$\frac{1.3.5.7.9}{2.4.6.8.10.11} = 380012893 \text{ negative}$$

Example XII.

$$\cos D = \sin (13^\circ 45' 13'' + 49513'').$$

$$10 \text{ times length of arc} = 2.40045818 + 2 \downarrow 1,8,7,6,0,3,1,5, \\ = \downarrow 87570364,$$

$$\begin{array}{r} 10^{\circ} \dots\dots - 230270081 \\ + 87570364 \\ \hline - 142699717, \text{ put} = x. \end{array}$$

$$\begin{array}{r} x^3 \dots\dots\dots - 428099151 \\ 1.2.3 \dots\dots\dots - 179184951 \\ \hline - 607284102 \\ 10^3 \dots\dots\dots + 690810243 \\ \hline 83526141 = 2 \downarrow 1,4,6,9,6,3,8,1, = 2.30532 \\ 69318201 \\ \hline 14207940 \\ 9531497 \\ \hline 4676443 \\ 3980332 \\ \hline 696111 \\ 599730 \\ \hline 9,6,3,8,1, \end{array}$$

$$\therefore \frac{x^3}{1.2.3} = .00230532, \text{ true to the last figure.}$$

$$\begin{array}{r} x^5 \dots\dots - 713498585 \\ 1.2.3.4.5 \dots\dots - 478773232 \\ \hline - 1192271817 \\ 10^5 + 1381620486 \\ \hline 189348669 = 6 \downarrow 1,0,6,3,2,4,9,1, = 6.64182 \end{array}$$

$$\therefore \frac{x^5}{1.2.3.4.5} = .00006642, \text{ true to the last figure.}$$

$$\begin{array}{r}
 x^7 \dots\dots - 998898019 \\
 1.2.3.4.5.6.7 \dots\dots - 852558978 \\
 \hline
 - 1851456997 \\
 10^9 + 2072430729 \\
 \hline
 220973732 = 9 \downarrow 0, 1, \dots\dots
 \end{array}$$

and will only give a unit in the eight decimal place.

$$\begin{array}{r}
 + \cdot 24004582 \\
 - \quad 230532 \\
 + \quad \quad 664 \\
 - \quad \quad \quad 1 \\
 \hline
 \cos 76^\circ 14' 47'' \dots \cdot 23774713 \quad \sin 13^\circ 45' 13''
 \end{array}$$

$$\begin{array}{r}
 A + B = 67^\circ \quad 1' \quad 37'' \\
 \quad \quad 90 \quad 0 \quad 0 \\
 \hline
 22^\circ 58' 23'' = 82703''; \text{ arc} = \cdot 400955459
 \end{array}$$

$$4\cdot 00955459 = 4 \downarrow 0, 0, 2, 3, 8, 6, 8, 3, = \downarrow \overline{138874995},$$

$$\begin{array}{r}
 10 \dots\dots - 230270081, \\
 + 138874995, \\
 \hline
 - 91395086, \text{ which put} = \overline{x}, 1
 \end{array}$$

$$\begin{array}{r}
 x^3 \dots\dots\dots - 274185258 \\
 1.2.3 \dots\dots\dots - 179184951 \\
 \hline
 - 453370209 \\
 10^2 \dots\dots\dots + 460540162 \\
 \hline
 7169953 = \downarrow 0, 7, 2, 0, 4, 4, 6, 2, = 1\cdot 07432862
 \end{array}$$

$$\therefore \frac{x^3}{1.2.3} = \cdot 01074329$$

$$\begin{array}{r}
 x^5 \dots - 456975430 \\
 1.2.3.4.5 \dots - 478773232 \\
 \hline
 - 935748662 \\
 10^5 \dots + 1151350405 \\
 \hline
 215601743 \\
 = 8 \downarrow 0, 7, 6, 8, 1, 8, 2, 8, = 8.63573736
 \end{array}$$

$$\therefore \frac{x^5}{1.2.3.4.5} = .00008636$$

$$\begin{array}{r}
 x^7 \dots - 639765602 \\
 1.2.3.4.5.6.7 \dots - 852558978 \\
 \hline
 - 1492324580 \\
 10^7 + 1611890567 \\
 \hline
 119565987 \\
 = 3 \downarrow 1, 0, 1, 6, 7, 7, 8, 5, = 3.3056 \dots
 \end{array}$$

$$\begin{array}{r}
 \therefore \frac{x^7}{1.2.3.4.5.6.7} = .00000033 \\
 + .40095546 \\
 - 1074329 \\
 + 8636 \\
 - 33 \\
 \hline
 .39029820 = \cos 67^\circ 1' 37''
 \end{array}$$

The next term will not give a unit in the eight decimal place.

$$\begin{array}{r}
 a + b \dots 90^\circ \quad 0' \quad 0'' \\
 \quad \quad 67 \quad 35 \quad 41 \\
 \hline
 22 \quad 24 \quad 19 = 80659''; .391045865 \text{ length.}
 \end{array}$$

$$\therefore .391045865 = 3 \downarrow 2, 7, 4, 7, 7, 1, 7, 4, = \downarrow \overline{136372319,}$$

$$\begin{array}{r}
 10 \dots - 230270081 \\
 \quad \quad + 136372319 \\
 \hline
 - 93897762 \text{ put} = \downarrow \overline{x, 1.} \\
 \quad \quad \quad \text{H H}
 \end{array}$$

$$\begin{array}{r}
 x^3 \dots\dots - 281693286 \\
 1.2.3 \dots - 179184951 \\
 \hline
 - 460878237 \\
 10^3 \dots\dots + 690810243 \\
 \hline
 229932006
 \end{array}$$

$$= 9 \downarrow 1,0,6,6,7,2,7,9, = 9.96625116$$

$$\begin{array}{r}
 x^5 \dots\dots - 469488810 \\
 1.2.3.4.5 \dots\dots - 478773232 \\
 \hline
 - 948262042 \\
 10^5 + 1151350405 \\
 \hline
 203088363
 \end{array}$$

$$= 7 \downarrow 0,8,5,2,7,1,2,9, = 7.6200397$$

$$\begin{array}{r}
 x^7 \dots - 657284334 \\
 1.2.3.4.5.6.7 \dots - 852558978 \\
 \hline
 10^7 \dots\dots - 1509843312 \\
 + 1611890567 \\
 \hline
 102047255
 \end{array}$$

$$= 2 \downarrow 3,4,1,5,4,2,7,6, = 4.775 \dots$$

$$\begin{array}{r}
 + 39104587 \dots \text{arc} \\
 - 996625 \\
 + 7620 \\
 28 \\
 \hline
 38115554 \cos 67^\circ 35' 41''.
 \end{array}$$

$$B = 20^\circ 25' 10'' = 73510'', \text{ length of arc} = .35638656$$

$$= \frac{1}{10} \downarrow 127090963,$$

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \dots$$

$$\begin{array}{r}
 - 230270081 \\
 + 127090963 \\
 \hline
 - 103179118 \text{ put} = \downarrow \overline{x},
 \end{array}$$

$$\begin{array}{r}
 x^2 \dots\dots\dots - 206358236 \\
 1.2 \dots\dots\dots - 69318201 \\
 \hline
 - 275676437 \\
 10^2 \dots\dots\dots + 460540162 \\
 \hline
 184863725
 \end{array}$$

$$= 6 \downarrow 0,5,7,0,3,6,7,4, = 6.35056866$$

$$\begin{array}{r}
 x^4 \dots\dots\dots - 412716472 \\
 1.2.3.4 \dots\dots\dots - 317821353 \\
 \hline
 - 730537825 \\
 10^4 \dots\dots\dots + 921080324 \\
 \hline
 190542499
 \end{array}$$

$$= 6 \downarrow 1,1,8,3,1,3,2,8, = 6.721619 \dots\dots$$

$$\begin{array}{r}
 x^6 \dots\dots\dots - 619074708 \\
 1.2.3.4.5.6 \dots\dots\dots - 657958183 \\
 \hline
 - 1277032891 \\
 10^6 \dots\dots\dots + 1381620486 \\
 \hline
 104587595
 \end{array}$$

$$= 2 \downarrow 3,6,7,0,4,7,3, = 2.8457 \dots\dots$$

$$\begin{array}{r}
 1.00000000 + \\
 .06350568 - \\
 .00067216 + \\
 .00000285 - \\
 \hline
 .93716363 \cos 20^\circ 25' 10''
 \end{array}$$

$$b = 20^\circ 22' 54'' = 73374''.$$

$$2 \mid 73374''00$$

$$\begin{array}{r} 36687 \ 00 \\ 1467 \ 48 \\ \hline \end{array}$$

$$\cdot 35219 \ 52 \downarrow 0,1,0,0,2,8,2,2, = \cdot 35572719 \text{ length of arc}$$

$$32 \mid \frac{3557271900}{111164747} = \downarrow 1,1,0,5,8,2,6,4, = \overline{10584844,}$$

$$\begin{array}{r} 10584844 \\ 32 = 2^5 \dots\dots 346591005 \\ \hline + 357175849 \\ 100 \dots\dots - 460540162 \\ \hline - 103364313, \mid = \downarrow x, \mid \\ - 206728626 \quad - 2 \\ - 413457252 \quad - 4 \\ - 620185878 \quad - 6 \end{array}$$

$$\begin{array}{r} x^2 \dots\dots - 206728626 \\ 1.2 \dots\dots - 69318201 \\ \hline - 276046827 \\ 10^2 \dots\dots + 460540162 \\ \hline 184493335 \\ = 6 \downarrow 0,5,3,3,3,1,0,4, = 6 \cdot 32709138 \end{array}$$

$$\begin{array}{r} x^4 \dots\dots\dots - 413457252 \\ 1.2.3.4 \dots\dots - 317821353 \\ \hline - 731278605 \\ 10^4 \dots\dots + 921080324 \\ \hline 189801719 \\ = 6 \downarrow 1,1,0,9,0,1,8,8, = 6 \cdot 672006 \dots \end{array}$$

$$\begin{array}{r} x^6 \dots\dots\dots - 620185878 \\ 1.2.3.4.5.6 \dots\dots - 657958183 \\ \hline - 1278144061 \\ 10^6 \dots\dots + 1381620486 \\ \hline 103476425 \\ = 2 \downarrow 3,5,5,8,8,5,4,3, = 2 \cdot 8142 \dots \end{array}$$

$$\begin{array}{r}
 1.00000000 + \\
 .06327091 - \\
 .00066720 + \\
 .00000281 - \\
 \hline
 \end{array}$$

$$.93739348 = \cos 20^\circ 22' 54''.$$

$$A = \begin{array}{ccc} 90^\circ & 0' & 0'' \\ 46 & 36 & 27 \end{array}$$

$$43 \quad 23 \quad 33 \text{ length of arc} = .757341994$$

$$.757341994 = 7 \downarrow 0, 7, 9, 0, 8, 6, 7, 8, = \downarrow \overline{202474649}, =$$

$$\begin{array}{r}
 + 202474649 \\
 10 \dots - 230270081 \\
 \hline
 - 27795432 = \downarrow x,
 \end{array}$$

$$\begin{array}{r}
 x^3 \dots - 83386296 \\
 1.2.3 \dots - 179184951 \\
 \hline
 - 262571247 \\
 10^2 \dots + 460540162 \\
 \hline
 197968915
 \end{array}$$

$$= 7 \downarrow 0, 3, 3, 8, 3, 0, 0, 6, = 7.2397717$$

$$\begin{array}{r}
 x^5 \dots - 138977160 \\
 1.2.3.4.5 \dots - 478773232 \\
 \hline
 - 617750392 \\
 10^3 \dots + 690810243 \\
 \hline
 73059851
 \end{array}$$

$$= 2 \downarrow 0, 3, 7, 5, 6, 7, 1, 6, = 2.07624672$$

$$\begin{array}{r}
 x^7 \dots - 194568024 \\
 1.2.3.4.5.6.7 \dots - 852558978 \\
 \hline
 - 1047127002 \\
 10^5 \dots + 1151350405 \\
 \hline
 104223403
 \end{array}$$

$$= 2 \downarrow 3, 6, 3, 4, 0, 3, 4, 8, = 2.8353 \dots$$

$$\begin{array}{r}
 x^9 \dots\dots - 250158888 \\
 1.2.3.4.5.6.7.8.9 \dots\dots - 1280247082 \\
 \hline
 - 1530405970 \\
 10^7 \dots\dots + 1611890567 \\
 \hline
 81484597
 \end{array}$$

$$= 2 \downarrow 1,2,6,4,5,0,0,3, = 2.258 \dots$$

$$\begin{array}{r}
 \text{Length of arc} = .75734199 + \\
 .07239772 - \\
 .00207625 + \\
 .00002835 - \\
 .00000023 + \\
 \hline
 .68699240 = \cos 46^\circ 36' 27''.
 \end{array}$$

$$\begin{array}{r}
 90^\circ \quad 0' \quad 0'' \\
 (a) \dots\dots 47 \quad 12 \quad 47 \\
 \hline
 42 \quad 47 \quad 13 = 154033'' \cdot 746773057 \text{ length of arc}
 \end{array}$$

$$7.46773057 = 7 \downarrow 0,6,4,9,8,1,0,6, = \downarrow \overline{201069219,}$$

$$\begin{array}{r}
 + 201069219 \\
 - 230270081 \\
 \hline
 - 29200862 = \downarrow x,
 \end{array}$$

$$\begin{array}{r}
 x^3 \dots\dots - 87602586 \\
 1.2.3 \dots\dots - 179184951 \\
 \hline
 - 266787537 \\
 10^2 \dots\dots + 460540162 \\
 \hline
 193752625
 \end{array}$$

$$= 6 \downarrow 1,5,0,6,0,7,6,2, = 6.94088166$$

$$\begin{array}{r}
 x^5 \dots\dots - 146004310 \\
 1.2.3.4.5 \dots\dots - 478773232 \\
 \hline
 - 624777542 \\
 10^3 \dots\dots + 690810243 \\
 \hline
 66032701
 \end{array}$$

$$= \downarrow 6,8,8,8,3,4,1,5, = 1.935361$$

$$\begin{array}{r}
 x^7 \dots\dots - 204406034 \\
 1.2.3.4.5.6.7 \dots\dots - 852558978 \\
 \hline
 - 1056965012 \\
 10^5 \dots\dots + 1151350405 \\
 \hline
 94385393 \\
 = 2 \downarrow 2,6,0,3,3,7,0,0, = 2.56974
 \end{array}$$

$$\begin{array}{r}
 x^8 \dots\dots\dots - 262807758 \\
 1.2.3.4.5.6.7.8.9 \dots\dots - 1280247082 \\
 \hline
 - 1543054840 \\
 10^7 \dots\dots + 1611890567 \\
 \hline
 68835727 \\
 = \downarrow 7,2,1,2,5,1,2,7, = 2.09 \dots
 \end{array}$$

$$\begin{array}{r}
 \text{Length of arc} = .74677306 + \\
 .06940882 - \\
 .00193536 + \\
 .00002570 - \\
 .00000021 + \\
 \hline
 .67927411 = \cos 47^\circ 12' 37''
 \end{array}$$

Example XIII.

When the term involving x^3 was operated upon by $\downarrow 5$, the term involving x^2 had to be operated upon by $\downarrow 3,3,1,9,1,9,6,2$, because

$$\begin{array}{r}
 9531497 = (A) \quad 995083 = (B) \quad 99955 = (C) \\
 \quad \quad \quad 5 \\
 \hline
 47657485 \\
 \quad \quad \quad 2 \\
 \hline
 3) 95314970 \\
 \hline
 31771657 \\
 28594491 = 3 (A) \\
 \hline
 3177166 \\
 2985249 = 3 (B) \\
 \hline
 191917 \\
 99955 = (C) \\
 \hline
 9,1,9,6,2
 \end{array}$$

∴ When ↓5, represents the cube of a number, ↓3,3,1,9,1,9,6,2, will represent the square of the same number.

Example XVI.

When the term affected by x^7 has to be operated upon by ↓7, ↓0,7, ↓0,0,7, &c. the co-efficient of x^6 must be operated upon by ↓6, ↓0,6, ↓0,0,6, &c.

When ↓0,0,3, becomes an operating figure for the seventh power, the operating figures for the third power are ↓0,0,1,2,8,5,5,9, but the number — 52 composed of two figures, is not altered by this a unit, and is therefore omitted in the operation.

In finding the consecutive numbers to the right of ↓, composing the root of an equation, it may often happen that the leading numbers of the divisor are destroyed, when the addition is performed. In such cases, it will be found convenient to transform the given equation into another equation. For example, take the equation

$$x^3 - 37.13394977 x^2 + 459.6430761 x = 1896.482019 (R).$$

and it will be found, that this equation may be verified by putting $x = 12.35843041$, $x = 12.3432103$, $x = 12.39455496$, or $x = 12.3809644$. Yet there is no guarantee that any one of these numbers is a root of the given equation. But there is no room for doubt when the given equation is verified by substituting

$$\downarrow 6,6,3,6,9,6,24,15, \text{ for } x^3; \downarrow 4,4,2,4,6,4,16,10, \text{ for } x^2;$$

and ↓2,2,1,2,3,2, 8, 5, for x ; Because

↓6,6,3,6,9,6,24,15, is evidently the cube of ↓2,2,1,2,3,2,8,5, and ↓4,4,2,4,6,4,16,10, is evidently the square of ↓2,2,1,2,3,2,8,5,

If 10 be substituted for x in the given equation, then

+ 3000	3 times	+ 1000·00
− 7426	twice	− 3713·39
+ 4596	once	+ 4596·43
* 170)		1883·04 take
		1896·48 from (<i>R</i>)
		13·44

* The leading figure of the divisor being destroyed, it is convenient to transform the given equation into another in which the leading figure of the divisor is not cancelled in the addition. Let the roots of the given equation be diminished by $\downarrow 2,2$,

$$\begin{array}{r}
 -3.713394977 \\
 \downarrow 2,2, = +1.234321 \\
 \hline
 -2.479073977 \text{ (p).} \quad (p) \downarrow 2,2, = -3.059973070364617 \\
 \downarrow 2,2, = +1.234321 \\
 \hline
 +1.244752977 \text{ (r).} \quad (r) \downarrow 2,2, = -1.536424739323617 \\
 \downarrow 2,2, = +1.234321 \\
 \hline
 -0.010431977 \\
 \hline
 \end{array}$$

$$(q) \downarrow 2,2, = \begin{array}{r} - 1.896482019 \\ + 1.896481993162757 \\ \hline .000000025837243 \end{array}$$

$\therefore x^3 - 10.431977x^2 + 32.9513117x = 25.837243$ is the transformed equation when its roots are multiplied by 1000. This last equation is easily operated upon, and roots found that may be relied upon. Further observations on the introductory examples are deemed unnecessary. The following examples illustrate a new method to find the sine, cosine, &c. of an arc without the use of tables, employment of impossible quantities, or the powers of the arc.

Let $\theta = \text{arc of } 20^\circ = .349065851 = .3 \downarrow 1, 5, 6, 4, 1, 9, 3$, then

$$\begin{array}{rcl}
 + 100000000 & \text{put} & = A \\
 - 34906585 & = A \div 1 \times 3 \downarrow 1,5,6, \dots & = B \\
 - 6092350 & = B \div 2 \times 3 \downarrow 1,5,6, \dots & = C \\
 - 708877 & = C \div 3 \times 3 \downarrow 1,5,6, \dots & = D \\
 + 61862 & = D \div 4 \times 3 \downarrow 1,5,6, \dots & = E \\
 + 4319 & = E \div 5 \times 3 \downarrow 1,5,6, \dots & = F \\
 - 252 & = F \div 6 \times 3 \downarrow 1,5,6, \dots & = G \\
 - 13 & = G \div 7 \times 3 \downarrow 1,5,6, \dots & = H
 \end{array}$$

$$\text{Hyp. log. } 1.41774258 = .34906585$$

$$\log (A + B + C + D + E + \dots) = \theta$$

$$\cos \theta = A - C + E - G$$

$$\sin \theta = B - D + F - H$$

$$\begin{array}{rcl}
 1.00000000 & + & = A \\
 6092350 & - & = C \\
 61862 & + & = E \\
 252 & - & = G
 \end{array}$$

$$\cos \theta = .93969260$$

$$\begin{array}{rcl}
 .34906585 & + & = B \\
 708877 & - & = D \\
 4319 & + & = F \\
 13 & - & = H
 \end{array}$$

$$\sin \theta = .34202014$$

The consecutive terms A, B, C, &c. are easily found, take for example, (F), the work unabridged will stand thus:

$$5 \overline{) 6186.2} \quad (E).$$

$$1237.2$$

$$3$$

$$3712$$

$$371$$

$$4083$$

$$204$$

$$4$$

$$4291 \dots$$

$$26 \dots$$

$$2 \dots$$

$$4319 = F.$$

The operating numbers employed being $\downarrow 1,5,6,4$, the remaining numbers are not required.

Let $\theta =$ an arc of $23^\circ 27' 25'' \cdot 42 = \cdot 409402949$

$$\cdot 409402949 = \cdot 4 \downarrow 0,2,3,3,3,6,1,8,$$

+ 1000000000		put	= A
+ 40940295	= A $\div$ 1 $\times$ $\cdot 4$	$\downarrow 0,2,3, \dots$	= B
- 8380537	= B $\div$ 2 $\times$ $\cdot 4$	$\downarrow 0,2,3, \dots$	= C
- 1143672	= C $\div$ 3 $\times$ $\cdot 4$	$\downarrow 0,2,3, \dots$	= D
+ 117054	= D $\div$ 4 $\times$ $\cdot 4$	$\downarrow 0,2,3, \dots$	= E
+ 9584	= E $\div$ 5 $\times$ $\cdot 4$	$\downarrow 0,2,3, \dots$	= F
- 654	= F $\div$ 6 $\times$ $\cdot 4$	$\downarrow 0,2,3, \dots$	= G
- 38	= G $\div$ 7 $\times$ $\cdot 4$	$\downarrow 0,2,3, \dots$	= H
+ 2	= H $\div$ 8 $\times$ $\cdot 4$	$\downarrow 0,2,3, \dots$	= I

$$\text{Hyp. log. of } 1\cdot 50591836 = \cdot 409402949 = \theta$$

1\cdot 00000000 + = A	\cdot 40940295 + = B
8380537 - = C	1143672 - = D
117054 + = E	9584 + = F
654 - = G	38 - = H
2 + = I	

$$\cos \theta = \cdot 91735865 \qquad \sin \theta = \cdot 39806169$$

Let $\theta =$ arc of $5^\circ 27' 39'' \cdot 14 = \cdot 09531018 = \cdot 09 \downarrow 0,5,7,5,8,$

+ 100000000		Put = A
+ 0953102	= A $\div$ 1 $\times$ $\cdot 09$	$\downarrow 0,5,7, \dots$
- 45420	= B $\div$ 2 $\times$ $\cdot 09$	$\downarrow 0,5,7, \dots$
- 1443	= C $\div$ 3 $\times$ $\cdot 09$	$\downarrow 0,5,7, \dots$
+ 34	= D $\div$ 4 $\times$ $\cdot 09$	$\downarrow 0,5,7, \dots$
+ 1	= E $\div$ 5 $\times$ $\cdot 09$	$\downarrow 0,5,7, \dots$

$$\text{Hyp. log. } 1\cdot 10000000 = \cdot 09531018$$

$$\text{Log. } (A + B + C + \dots) = \theta.$$

$$\cos \theta = A - C + E.$$

$$\sin \theta = B - D + F.$$

$$\begin{array}{r} 1.0000000 + = A \\ 45420 - = C \\ 34 + = E \end{array}$$

$$\begin{array}{r} .0953102 + = B \\ 1443 - = D \\ 1 + = F \end{array}$$

$$\cos \theta = .9954614$$

$$\sin \theta = .0951660$$

The quantities C, D, and E, have only to be determined, for A and B are given, and it is easily observed that $F = 1$. The following unabridged work shows that C, D, and E, are found in a few minutes.

$$\begin{array}{r} 4765.51 \\ \hline 9 \end{array}$$

$$\begin{array}{r} 42 \overline{) 890} . \\ 2 \overline{) 145} . \\ \hline 4 \overline{) 3} . \end{array}$$

$$\begin{array}{r} 450 \overline{) 78} \\ \cdot 3 \overline{) 15} \\ 23 \\ \hline 4 \end{array}$$

$$45420 = C$$

$$\begin{array}{r} 454.20 \\ \hline 3 \end{array}$$

$$\begin{array}{r} 13 \overline{) 63} \\ \hline 68 \\ \hline 1 \end{array}$$

$$\begin{array}{r} 143 \overline{) 2} . . \\ 1 \overline{) 0} . . \\ \hline 1 . . \end{array}$$

$$1443 = D$$

$$4 \overline{) 14.43}$$

$$3.61$$

$$9$$

$$\begin{array}{r} 32 \overline{) .} . \\ 2 \overline{) .} . \end{array}$$

$$34 = F$$

Given $\theta = .225154191074$ an arc of $12^\circ 54' 1'' .3855975$, to find $\sin \theta$ and $\cos \theta$ to twelve places of decimals.

$$\theta = .2 \downarrow 1,2,3,2,5,8,7,6,2,3,5,$$

$$\begin{array}{rcl} + 10000000000000 & = & = A \\ + 225154191074 & = & A \div 1 \times .2 \downarrow 1,2,3, \dots = B \\ - 25347204876 & = & B \div 2 \times .2 \downarrow 1,2,3, \dots = C \\ - 1902443160 & = & C \div 3 \times .2 \downarrow 1,2,3, \dots = D \\ + 107085855 & = & D \div 4 \times .2 \downarrow 1,2,3, \dots = E \\ + 4822150 & = & E \div 5 \times .2 \downarrow 1,2,3, \dots = F \\ - 180953 & = & F \div 6 \times .2 \downarrow 1,2,3, \dots = G \\ - 5820 & = & G \div 7 \times .2 \downarrow 1,2,3, \dots = H \\ + 164 & = & H \div 8 \times .2 \downarrow 1,2,3, \dots = I \\ + 4 & = & I \div 9 \times .2 \downarrow 1,2,3, \dots = J \end{array}$$

$$\log. 1.252515934056 = \theta.$$

$$\text{Log. } (A + B + C + \dots) = \theta.$$

$$A - C + E - G + I = \cos \theta.$$

$$B - D + F - H + J = \sin \theta.$$

$$1.00000000000000 + = A$$

$$25347204876 - = C$$

$$107085855 + = E$$

$$180953 - = G$$

$$164 + = I$$

$$\underline{} \\ .974759700190 = \cos \theta.$$

$$.225154191074 + = B$$

$$1902443160 - = D$$

$$4822150 + = F$$

$$5820 - = H$$

$$4 + = J$$

$$\underline{} \\ .223256564248 = \sin \theta.$$

The operative figures $\downarrow 1, 2, 3, 2, 5, \dots$ fall out of use, one by one, as the results C, D, E, &c. decrease; for example, in finding H, only $\downarrow 1, 2, 3, 2$, are required.

$$G = 18095.3$$

$$7 \overline{) 36191.}$$

$$\begin{array}{r} 5 \overline{) 170} \\ 5 \overline{) 17} \end{array}$$

$$\begin{array}{r} 5 \overline{) 687} \dots \\ 1 \overline{) 14} \dots \\ 1 \overline{) \dots} \end{array}$$

$$\begin{array}{r} 5 \overline{) 802} \dots \\ 1 \overline{) 7} \dots \end{array}$$

$$\begin{array}{r} 5 \overline{) 819} \dots \\ 1 \overline{) \dots} \end{array}$$

$$5820 = H.$$

The sum of the series

$$\frac{2}{\sqrt{\pi}} \left(x - \frac{1}{1.3} x^3 + \frac{1}{1.25} x^5 - \frac{1}{1.237} x^7 + \frac{1}{1.2349} x^9 - \frac{1}{1.234511} x^{11} + \dots \right)$$

may be found in a similar manner (see Example 10, page xxi).

$$\frac{2}{\sqrt{\pi}} x = \downarrow 1, 1, 8, 5, 7, 2, 8, 5, = 1.12056087$$

$$x^2 = .9 \downarrow 0, 9, 1, 9, 1, 2, 3, \text{ which put } = .9 \downarrow u, \dots$$

$$\begin{array}{rcl}
1'12056087 & + = A & \\
\cdot 36836583 & - = A \times \frac{1 \times .9}{1 \times 3} & \downarrow u_1 \dots = B \\
\cdot 10898475 & + = B \times \frac{3 \times .9}{2 \times 5} & \downarrow u_1 \dots = C \\
2559080 & - = C \times \frac{5 \times .9}{3 \times 7} & \downarrow u_1 \dots = D \\
490734 & + = D \times \frac{7 \times .9}{4 \times 9} & \downarrow u_1 \dots = E \\
79194 & - = E \times \frac{9 \times .9}{5 \times 11} & \downarrow u_1 \dots = F \\
11016 & + = F \times \frac{11 \times .9}{6 \times 13} & \downarrow u_1 \dots = G \\
1342 & - = G \times \frac{13 \times .9}{7 \times 15} & \downarrow u_1 \dots = H \\
146 & + = H \times \frac{15 \times .9}{8 \times 17} & \downarrow u_1 \dots = I \\
13 & - = I \times \frac{17 \times .9}{9 \times 19} & \downarrow u_1 \dots = J \\
1 & + = J \times \frac{19 \times .9}{10 \times 21} & \downarrow u_1 \dots = K
\end{array}$$

$\cdot 83980247$, area of the curve

x^2 and x may be developed under many forms, and the series summed in a similar manner; for example, x^2 may be put $= .8 \downarrow 2, 1, 8, 6, 7, 1, 2, 8$, and the same result obtained.

THE END.

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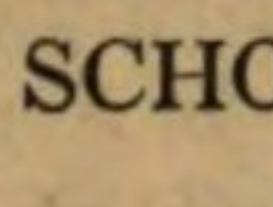
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